

ADVANCES IN CARDINAL ARITHMETIC

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ANNOTATED CONTENT

§1 $I[\lambda]$ is quite large

[If $\text{cf}\kappa = \kappa, \kappa^+ < \text{cf}\lambda = \lambda$ then there is a stationary subset S of $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ in $I[\lambda]$. Moreover, we can find $\bar{C} = \langle C_\delta : \delta \in S \rangle$, C_δ a club of λ , $\text{otp}(C_\delta) = \kappa$, guessing clubs and for each $\alpha < \lambda$ we have: $\{C_\delta \cap \alpha : \alpha \in \text{nacc } C_\delta\}$ has cardinality $< \lambda$.]

§2 Measuring $\mathcal{S}_{<\kappa}(\lambda)$

[We prove that e.g. there is a stationary subset of $\mathcal{S}_{<\aleph_1}(\lambda)$ of cardinality $\text{cf}(\mathcal{S}_{<\aleph_1}(\lambda), \subseteq)$.]

§3 Nice filters revisited

[We prove the existence of nice filters when instead being normal filters on ω_1 they are normal filters with larger domains, which can increase during a play. They can help us transfer situation on \aleph_1 -complete filters to normal ones].

§4 Ranks

[We reconsider ranks and niceness of normal filters, such that we can pass say from $pp_{\Gamma(\aleph_1)}(\mu)$ (where $\text{cf}\mu = \aleph_1$) to $pp_{\text{normal}}(\mu)$.]

§5 More on ranks and higher objects

§6 Hypotheses

[We consider some weakenings of G.C.H. and their consequences. Most have not been proved independent of ZFC.]

§1 $I[\lambda]$ IS QUITE LARGE AND GUESSING CLUBS

On $I[\lambda]$ see [Sh 108], [Sh 88a], [Sh 351, §4] (but this section is self-contained; see Definition 1.1 and Claim 1.3 below). We shall prove that for regular κ, λ , such that $\kappa^+ < \lambda$, there is a stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ in $I[\lambda]$. We then investigate “guessing clubs” in (ZFC).

1.1 Definition. For a regular uncountable cardinal λ , $I[\lambda]$ is the family of $A \subseteq \lambda$ such that $\{\delta \in A : \delta = \text{cf}(\delta)\}$ is not stationary and for some $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ we have:

- (a) \mathcal{P}_α is a family of $< \lambda$ subsets of α
- (b) for every limit $\alpha \in A$ of cofinality $< \alpha$ there is $x \subseteq \alpha$, $\text{otp}(x) < \alpha = \sup(x)$ such that $\zeta < \alpha \Rightarrow x \cap \zeta \in \{\mathcal{P}_\gamma : \gamma < \alpha\}$.

1.2 Observation. In Definition 1.1 we can weaken (b) to:

for some club E of x for every limit $\alpha \in A \cap E$ of cofinality $< \alpha \dots$

Proof. Just replace \mathcal{P}_α by $\{x \cap \alpha : x \in \cup\{\mathcal{P}_\beta : \beta \leq \text{Min}(E \setminus \alpha + 1)\}\}$.

We know (see [Sh 108], [Sh 88a] or below)

1.3 Claim. Let $\lambda > \aleph_0$ be regular.

1) $A \in I[\lambda]$ iff (note: by (c) below the set of inaccessible in A is not stationary and) there is $\langle C_\alpha : \alpha < \lambda \rangle$ such that:

- (a) C_α is a closed subset of α
- (b) if $\alpha^* \in \text{nacc}(C_\alpha)$ then $C_{\alpha^*} = C_\alpha \cap \alpha$ (nacc stands for “non-accumulation”)
- (c) for some club E of λ , for every $\delta \in A \cap E$, we have: $\text{cf}(\delta) < \delta$ and $\delta = \sup(C_\delta)$, and $\text{cf}(\delta) = \text{otp}(C_\delta)$
- (d) $\text{nacc}(C_\alpha)$ is a set of successor ordinals.

2) $I[\lambda]$ is a normal ideal.

Proof. 1) The “if” part:

Assume $\langle C_\beta : \beta < \lambda \rangle$ satisfy (a), (b), (c) with a club E for (c). For each limit $\alpha < \lambda$ choose a club e_α of order type $\text{cf}(\alpha)$. We define, for $\alpha < \lambda$:

$$\mathcal{P}_\alpha =: \{C_\beta : \beta \leq \alpha\} \cup \{e_\beta : \beta \leq \alpha\} \cup \{e_\gamma \cap \alpha : \gamma \leq \text{Min}(E \setminus (\alpha + 1))\}.$$

It is easy to check that $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ exemplify “ $A \in I[\lambda]$ ”.

The “only if” part:

Let $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ exemplify “ $A \in I[\lambda]$ ” (by Definition 1.1). Without loss of generality

(*) if $C \in \mathcal{P}_\alpha$, and $\zeta \in C$ then $C \setminus \zeta \in \mathcal{P}_\alpha$ and $C \cap \zeta \in \mathcal{P}_\alpha$

For each limit $\beta < \lambda$ let e_β be a club of β satisfying $\text{otp}(e_\beta) = \text{cf}(\beta)$ and $\text{cf}(\beta) < \beta \Rightarrow \text{cf}(\beta) < \min(e_\beta)$. Let $\langle \gamma_i : i < \lambda \rangle$ be strictly increasing continuous, each γ_i a non-successor ordinal $< \lambda$, $\gamma_0 = 0$, and $\gamma_{i+1} - \gamma_i \geq \aleph_0 + |\bigcup_{\alpha \leq \gamma_i} \mathcal{P}_\alpha| + |\gamma_i|$

and $\gamma_i \in A \Rightarrow \text{cf}(\gamma_i) < \gamma_i$.

(Why? Let E' be a club of λ such that $\gamma \in E \cap A \Rightarrow \text{cf}(\gamma) < \gamma$, and then choose $\gamma_i \in E$ by induction on $i < \lambda$.)

Let F_i be a one to one function from $(\bigcup_{\alpha \leq \gamma_i} \mathcal{P}_\alpha) \times \gamma_i$ into $\{\zeta + 1 : \gamma_i < \zeta + 1 < \gamma_{i+1}\}$.

Now we choose $C_\alpha \subseteq \alpha$ as follows. First, for $\aleph = 0$ let $C_\alpha = \emptyset$. Second, assume α is a successor ordinal, let $i(\alpha)$ be such that $\gamma_{i(\alpha)} < \alpha < \gamma_{i(\alpha)+1}$. If $\alpha \notin \text{Rang}(F_{i(\alpha)})$, let $C_\alpha = \emptyset$. If $\alpha = F_{i(\alpha)}(x, \beta)$ hence necessarily $x \in \bigcup_{\epsilon \leq \gamma_{i(\alpha)}} \mathcal{P}_\epsilon$, $\beta < \gamma_{i(\alpha)}$ and x, β

are unique. Let C_α be the closure (in the order topology) of C_α^- , which is defined as:

$$\{F_j(x \cap \zeta, \beta) : \text{the sequence } (j, \zeta, \beta) \text{ satisfies } (*)_{j, \zeta}^{x, \beta} \text{ below}\} \text{ where}$$

$$\boxtimes_{j, \zeta}^{x, \beta}(i) \quad \zeta \in x$$

$$(ii) \quad \text{otp}(x \cap \zeta) \in e_\beta,$$

$$(iii) \quad j < i(\alpha) \text{ is minimal such that } x \cap \zeta \in \bigcup_{\epsilon \leq \gamma_j} \mathcal{P}_\epsilon$$

$$(iv) \quad \text{if } \xi \in x \cap \zeta, \text{ otp}(x \cap \xi) \in e_\beta \text{ then} \\ (\exists j(1) < j)[x \cap \xi \in \bigcup_{\epsilon \leq \gamma_{j(1)}} \mathcal{P}_\epsilon]$$

$$(v) \quad \beta < \text{Min}(x).$$

Third, for $\alpha < \lambda$ limit, choose C_α : if possible, $\text{nacc}(C_\alpha)$ is a set of successor ordinals, C_α is a club of α , $[\beta \in \text{nacc}(C_\alpha) \Rightarrow C_\beta = \beta \cap C_\alpha]$; if this is impossible, let $C_\delta = \emptyset$. Lastly, let $C_0 = \emptyset$ and let $E =: \{\gamma_i : i \text{ is a limit ordinal } < \lambda\}$. Now we can check the condition in 1.3(1).

Note that for α successor $C_\alpha^- = \text{nacc}(C_\alpha)$.

Clause (a): C_α a closed subset of α .

If $\alpha = 0$ trivial as $C_\alpha = \emptyset$ and if α is a limit ordinal, this is immediate by the definition. So let α be a successor ordinal, hence, by the choice of $\langle \gamma_i : i < \lambda \rangle$ as an increasing continuous sequence of nonsuccessor ordinals with $\gamma_0 = 0$, clearly $i(\alpha)$ is well defined, $\gamma_{i(\alpha)} < \alpha < \gamma_{i(\alpha)+1}$. Now if $\alpha \notin \text{Rang}(F_{i(\alpha)})$ then $C_\alpha = \emptyset$ and we are done so for some x, β we have $\alpha = F_{i(\alpha)}(x, \beta)$ hence necessarily $x \in \bigcup_{\epsilon \leq \gamma_{i(\alpha)}} \mathcal{P}_\epsilon$ and

$\beta < \gamma_{i(\alpha)}$. By the definition of C_α (the closure in the order topology on α , of the set of C_α^- i.e. the set of $F_j(x \cap \zeta, \beta)$ for the pair (j, ζ) satisfying $\boxtimes_{j, \zeta}^{x, \beta}$ it suffices to show $C_\alpha^- \subseteq \alpha$, i.e.

(*) if the pair (j, ζ) satisfies $\boxtimes_{j, \zeta}^{x, \beta}$ then $F_j(x \cap \zeta, \beta) < \alpha$.

So assume (j, ζ) satisfies $\boxtimes_{j, \zeta}^{x, \beta}$ but by clause (iii) we know that $j < i(\alpha)$ and so $\text{Rang}(F_j) \subseteq \gamma_{j+1} \subseteq \gamma_{i(\alpha)} < \alpha$ as required.

Clause (b): If $\alpha^* \in \text{nacc}(C_\alpha)$ then $C_{\alpha^*} = C_\alpha \cap \alpha^*$.

If it is enough to show $C_{\alpha^*}^- = \alpha^* \cap C_\alpha^-$ and as $C_\alpha^- = \text{nacc}(C_\alpha)$, we have $\alpha^* \in C_\alpha^-$. As $\alpha^* \in C_\alpha^-$ necessarily for some ζ, j satisfying $\boxtimes_{j, \zeta}^{x, \beta}$ we have $\alpha^* = F_j(x \cap \zeta, \beta)$. By the choice of F_j necessarily α^* is a successor ordinal and $\gamma_j < \alpha^* < \gamma_{j+1}$.

Now any member $\alpha(1)$ of $\alpha^* \cap C_\alpha^-$ has the form $F_{j(1)}(x \cap \zeta(1), \beta)$ with $j(1), \zeta(1)$ satisfying $\boxtimes_{j, \zeta}^{x, \beta}$; clearly $\gamma_{j(1)} < \alpha(1) = F_{j(*)}(x \cap \zeta(1), \beta) < \gamma_{j(1)+1}$ and $\gamma_j < \alpha^* = F_j(x \cap \zeta, \beta) < \gamma_{j+1}$. But $\alpha(1) < \alpha^*$ (being in $\alpha^* \cap C_\alpha^-$) so necessarily $j(1) + 1 \leq j$. So $j(1), \zeta(1)$ satisfy (i) – (v) with x replaced by $x \cap \zeta$, i.e., satisfy $\boxtimes_{j, \zeta}^{x, \beta}$; recall by $\alpha^* = F_j(x \cap \zeta, \beta)$, so $F_{j(x)}(x \cap \zeta(1), \beta) \in C_{\alpha^*}^-$. So $\alpha^* \cap C_\alpha^- \subseteq C_{\alpha^*}^-$; similarly $C_{\alpha^*}^- \subseteq \alpha^* \cap C_\alpha^-$, so we get the desired equality.

Clause (c): We shall show that $E = \{\gamma_i : i \text{ is a limit ordinal } < \lambda\}$ is as required in closed (c).

Clearly E is a club of λ . So assume that $\delta \in A \cap E$ we should prove: $\text{cf}(\delta) < \delta$, $\delta = \sup(C_\delta)$, $\text{cf}(\delta) = \text{otp}(C_\delta)$. Now $\delta \in E \cap A \Rightarrow \delta > \text{cf}(\delta)$ holds as we assume $\gamma_i \in A \Rightarrow \text{cf}(\gamma_i) < \gamma_i$. As $\delta \in E$, by E 's definition for some limit ordinal $i(*)$ we have $\delta = \gamma_{i(*)}$. By the choice of C_δ it is enough to find a set C closed unbounded in δ of order type $\text{cf}(\delta)$ such that $\alpha \in \text{nacc}(C) \Rightarrow \alpha$ successor & $C_\alpha = C \cap \alpha$.

By the choice of $\bar{\mathcal{P}}$, for some $x \subseteq \delta$, $\text{otp}(x) < \delta = \sup(x)$ and $\bigwedge_{\zeta < \delta} x \cap \zeta \in \bigcup_{\gamma < \delta} \mathcal{P}_\gamma$.

By (*) above also $\xi \in x$ & $\bar{S} \in x \setminus \xi \Rightarrow x \cap \zeta \setminus \xi \in \bigcup_{\gamma < \delta} \mathcal{P}_\gamma$ so without loss of generality $\text{otp}(x) < \text{Min}(x)$. Let $\beta = \text{otp}(x)$, so we know that β is a limit ordinal, moreover $\text{cf}(\beta) = \text{cf}(\delta)$. Remember e_β is a club of β of order type $\text{cf}(\beta)$ which is $\text{cf}(\delta)$. Let

$$y =: \{\zeta \in x : \text{otp}(x \cap \zeta) \in e_\beta\}.$$

Clearly y is a subset of x of order type $\text{otp}(e_\beta) = \text{cf}(\delta)$. Define $h : y \rightarrow i(*)$ by $h(\zeta) = \text{Min}\{j : x \cap \zeta \in \bigcup_{\epsilon \leq \gamma_j} \mathcal{P}_\epsilon\}$, so by (*) we know that h is non-decreasing, and

by the choice of x , $\bigwedge_{\zeta \in y} \gamma_{h(\zeta)} < \delta$, equivalently $\bigwedge_{\zeta \in y} h(\zeta) < i(*)$.

Let $z = \{\zeta \in y : \text{for every } \xi \in y \cap \zeta \text{ we have } h(\xi) < h(\zeta)\}$. Let $C^- = \{F_{h(\zeta)}(x \cap \zeta, \beta) : \zeta \in z\}$; it satisfies: $C^- \subseteq \delta = \sup^\alpha \delta_\alpha$ and it is easy to check, as in the proof of clause (c) that $[\alpha \in C^- \Rightarrow C_\alpha^- = C^- \cap \alpha]$. So by the choice of C^- its closure in δ is as required.

Clause (d): $\text{nacc}(C_\alpha)$ is a set of successor ordinals.

Check.

Remark. 1) We could also strengthen (*) to make $z \cap \zeta \in \mathcal{P}_{h(\zeta)}$.

2) By Definition 1.1 we know that $I[\lambda]$ is an ideal; by 1.3(1) we know that $I[\lambda]$ includes the ideal of non-stationary subsets of λ . By the last phrase and Definition 1.1, clearly $I[\lambda]$ is normal. $\square_{1.3}$

1.4 Claim. *If κ, λ are regular, $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$, $S \in I[\lambda]$, S stationary, $\kappa^+ < \lambda$ then we can find $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ such that for $\delta(*) =: \kappa$ we have:*

$\oplus_{\mathcal{P}_S}^{\lambda, \delta(*)} (i)$ \mathcal{P}_α is a family of closed subsets of α , $|\mathcal{P}_\alpha| < \lambda$

(ii) $\text{otp}(C) \leq \delta(*)$ for $C \in \bigcup_{\alpha} \mathcal{P}_\alpha$

(iii) for some club E of λ , we have:

$[\alpha \notin E \Rightarrow \mathcal{P}_\alpha = \emptyset]$ and

$[\alpha \in E \Rightarrow (\forall C \in \mathcal{P}_\alpha)(\text{otp}(C) \leq \delta(*))]$

$[\alpha \in E \setminus (S \cap \text{acc}(E)) \Rightarrow (\forall C \in \mathcal{P}_\alpha)[\text{otp}(C) < \delta(*)]$

$[\alpha \in S \cap \text{acc}(E) \Rightarrow (\exists! C \in \mathcal{P}_\alpha)(\text{otp}(C) = \delta(*))]$

$[\alpha \in S \cap \text{acc}(E) \text{ \& } C \in \mathcal{P}_\alpha \text{ \& } \text{otp}(C) = \delta(*) \Rightarrow \alpha = \sup(C))]$

- (iv) $C \in \mathcal{P}_\alpha$ & $\beta \in \text{nacc}(C) \Rightarrow \beta \cap C \in \mathcal{P}_\beta$
- (v) for any club E' of λ for some $\delta \in S \cap E'$ and $C \in \mathcal{P}_\delta$ we have $C \subseteq E'$ & $\text{otp}(C) = \delta(*)$.

Proof. Let $\langle C_\alpha : \alpha < \lambda \rangle$ witness “ $S \in I[\lambda]$ ” be as in 1.3(1); without loss of generality $\text{otp}(C_\alpha) \leq \delta(*)$. For any club E , consisting of limit ordinals for simplicity, let us define \mathcal{P}_E^α by induction on $\alpha < \lambda$:

$$\begin{aligned} \mathcal{P}_E^\alpha = & \{ \alpha \cap \text{gl}(C_\beta, E) : \alpha \in E \text{ and } \alpha \leq \beta < \text{Min}[E \setminus (\alpha + 1)] \} \\ & \cup \{ C \cup \{ \beta \} : \beta \in E \cap \alpha, C \in \mathcal{P}_E^\beta \text{ and } \text{otp}(C) < \delta(*) \} \end{aligned}$$

where

$$\text{gl}(C_\beta, E) = \{ \sup(E \cap (\gamma + 1)) : \gamma \in C_\beta \text{ and } \gamma > \text{Min}(E) \}.$$

Note that $|\mathcal{P}_E^\alpha| \leq |\text{Min}(E \setminus (\alpha + 1))| < \lambda$.

We can prove that for some club E of λ the sequence $\langle \mathcal{P}_E^\alpha : \alpha < \lambda \rangle$ is as required except possibly clause (v) which can be corrected gotten by a right of E (just by trying successively κ^+ clubs E_ζ (for $\zeta < \kappa^+$) decreasing with ζ , see [Sh 365]). Note that clause (iv) guaranteed by demanding E to consist of limit ordinals only and the second set in the union defining \mathcal{P}_E^α . $\square_{1.4}$

The following lemma gives sufficient condition for the existence of “quite large” stationary sets in $I[\lambda]$ of almost any fixed cofinality.

1.5 Lemma. *Suppose*

- (i) $\lambda > \kappa > \aleph_0$, λ and κ are regular
- (ii) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$, \mathcal{P}_α a family of $< \lambda$ closed subsets of α
- (iii) $I_{\bar{\mathcal{P}}} = \{ S \subseteq \kappa : \text{for some club } E \text{ of } \kappa \text{ for no } \delta \in S \cap E \text{ is there a club } C \text{ of } \delta, \text{ such that } C \subseteq E \text{ and } [\alpha \in \text{nacc}(C) \Rightarrow C \cap \alpha \in \bigcup_{\beta < \alpha} \mathcal{P}_\beta] \}$ is a proper ideal on κ .

Then there is $S^* \in I[\lambda]$ such that for stationarily many $\delta < \lambda$ of cofinality κ , $S^* \cap \delta$ is stationary in δ , moreover for some club E of δ of order type κ

$$\{ \text{otp}(\alpha \cap E) : \alpha \in E \setminus S^* \} \in I_{\bar{\mathcal{P}}}.$$

- 1.6 Remark. 1) The “for stationarily many” in the conclusion can be strengthened to: a set whose complement is in the ideal defined in [Sh 371, §2].
 2) So if $\kappa^\sigma < \lambda$ then we can have $\{i < \kappa : \text{cf}(i) = \sigma\} \in I_{\bar{\mathcal{P}}}$.

Proof. Let χ be regular large enough, N^* be an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi^*)$ of cardinality λ such that $(\lambda + 1) \subseteq N^*$, $\bar{\mathcal{P}} \in N$. Let $\bar{C} = \langle C_i : i < \lambda \rangle$ list $N^* \cap \{A \subseteq \lambda : |A| < \kappa\}$ and let

$$S^* = \{\delta < \lambda : \text{cf}(\delta) < \kappa \text{ and for some } A \subseteq \delta \text{ satisfying } \delta = \sup(A), \text{ we have } \text{otp}(A) < \kappa \text{ and } (\forall \alpha < \delta)[A \cap \alpha \in \{C_i : i < \delta\}]\}.$$

Clearly $S^* \in I[\lambda]$; so we should only find enough $\delta < \lambda$ of cofinality κ as required in the conclusion of 1.5. So let E^* be a club of λ and we shall prove that such $\delta \in E^*$ exists. We can choose M_ζ by induction on $\zeta \leq \kappa$ such that:

- (a) $M_\zeta \prec (\mathcal{H}(\chi), \in, <_\chi^*)$
- (b) $\|M_\zeta\| < \lambda$, $M_\zeta \cap \lambda$ an ordinal
- (c) M_ζ is increasing continuous
- (d) $N, \kappa, \bar{\mathcal{P}}, \bar{C}, E^*$ belongs to M_0
- (e) $\langle M_\epsilon : \epsilon \leq \zeta \rangle \in M_{\zeta+1}$.

Let $\delta_\zeta = \sup(M_\zeta \cap \lambda)$, clearly $\delta_\zeta \in E^*$ for every $\zeta \leq \kappa$ and $\langle \delta_\zeta : \zeta \leq \kappa \rangle$ is a (strictly) increasing continuous, so $\delta =: \delta_\kappa$ has cofinality κ . Hence there is a (strictly) increasing continuous sequence $\langle \alpha_\zeta : \zeta < \kappa \rangle \in N^*$ with limit δ , and clearly $E = \{\zeta < \kappa : \alpha_\zeta = \delta_\zeta \text{ and } \zeta \text{ is a limit ordinal}\}$ is a club of κ . We know that

$$T =: \{\zeta < \kappa : \zeta \in E \text{ and for some club } C \text{ of } \zeta, C \subseteq E \text{ and } \bigwedge_{\epsilon < \zeta} [C \cap \epsilon \in \bigcup_{\xi < \zeta} \mathcal{P}_\xi]\}.$$

is stationary; moreover, $\kappa \setminus T \in I_{\bar{\mathcal{P}}}$ (see assumption (iii)) and clearly $T \subseteq E$. Clearly it suffices to show

$$(*) \quad \zeta \in T \Rightarrow \delta_\zeta \in S^*.$$

Suppose $\zeta \in T$, so there is C , a club of ζ such that $C \subseteq E$ and $\bigwedge_{\epsilon < \zeta} [C \cap \epsilon \in \bigcup_{\xi < \zeta} \mathcal{P}_\xi]$.

Let $C^* = \{\delta_\epsilon : \epsilon \in C\}$, so C^* is a club of δ_ζ of order type $\leq \zeta < \kappa$ (which is $< \delta_0 \leq \delta_\zeta$). It suffices to show for $\xi \in C$ that $\{\delta_\epsilon : \epsilon \in \xi \cap C\} \in \{C_i : i < \delta_\zeta\}$.

For this end we shall show

$$(\alpha) \quad \{\delta_\epsilon : \epsilon \in C \cap \xi\} \in \{C_i : i < \lambda\}$$

$$(\beta) \quad \{\delta_\epsilon : \epsilon \in C \cap \xi\} \in M_{\xi+1}.$$

This suffices as $\langle C_i : i < \lambda \rangle \in M_0 \prec M_{\xi+1}$ and $M_{\xi+1} \cap \{C_i : i < \lambda\} = \{C_i : i \in \lambda \cap M_{\xi+1}\} = \{C_i : i < \delta_{\xi+1}\}$.

Proof of (α) . Remember $\langle \alpha_\epsilon : \epsilon < \kappa \rangle \in N^*$. Also $\bar{\mathcal{P}} = \langle \mathcal{P}_\epsilon : \epsilon < \kappa \rangle \in N^*$ hence $\bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon \subseteq N^*$ (as $\kappa < \lambda, |\mathcal{P}_\epsilon| < \lambda, \lambda + 1 \subseteq N, \bar{\mathcal{P}} \in N^*$ so now for $\xi \in C$ we

have $C \cap \xi \in \bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon$; hence $C \cap \xi \in N^*$. Together $\{\alpha_\epsilon : \epsilon \in \xi \cap C\} \in N^*$; as $\epsilon \in C \Rightarrow \epsilon \in E \Rightarrow \alpha_\epsilon = \delta_\epsilon$ (as $C \subseteq E$ and the definition of E), and the definition of $\langle C_i : i < \lambda \rangle$, we are done.

Proof of (β) . We know $\bar{\mathcal{P}} \in M_0$; as $|\mathcal{P}_\epsilon| < \lambda, \kappa < \lambda$ clearly $|\bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon| < \lambda$ so as

$M_\epsilon \cap \lambda$ is an ordinal, clearly $\bigcup_{\epsilon < \kappa} \mathcal{P}_\epsilon \subseteq M_0$. So for $\epsilon < \zeta$ we have $C \cap \epsilon \in \bigcup_{\gamma < \zeta} \mathcal{P}_\gamma \subseteq$

$M_0 \subseteq M_{\xi+1}$. As $\langle M_i : i \leq \xi \rangle \in M_{\xi+1}$ clearly $\langle \delta_i : i \leq \xi \rangle \in M_{\xi+1}$ hence by the previous sentence also $\langle \delta_i : i \in C \cap \xi \rangle \in M_{\xi+1}$, as required. $\square_{1.5}$

1.7 Conclusion. If κ, λ are regular, $\kappa^+ < \lambda$ then there is a stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ in $I[\lambda]$.

Proof. If $\lambda = \kappa^{++}$ - use [Sh 351, 4.1]. So assume $\lambda > \kappa^{++}$. By [Sh 351, 4.1] the pair (κ, κ^{++}) satisfies the assumption of 1.4 for $S = \{\delta < \kappa^{++} : \text{cf}(\delta) = \kappa\}$; (i.e. κ, λ there stands for κ, κ^{++} here). Hence the conclusion of 1.4 holds for some $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \kappa^{++} \rangle, |\mathcal{P}_\alpha| < \kappa^{++}$. Now apply 1.5 with (κ^{++}, λ) here standing for (κ, λ) there (we have just proved $I_{\bar{\mathcal{P}}}$ is a proper ideal, so assumption (ii) holds). Note:

$$(*) \quad \{\delta < \kappa^{++} : \text{cf}(\delta) = \kappa\} \notin I_{\bar{\mathcal{P}}}.$$

Now the conclusion of 1.5 (see the moreover and choice of $\bar{\mathcal{P}}$ i.e. $(*)$) gives the desired conclusion. $\square_{1.7}$

1.8 Conclusion. If $\lambda > \kappa$ are uncountable regular, $\kappa^+ < \lambda$, then for some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ and some $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ we have: $\oplus_{\bar{\mathcal{P}}, S}^{\lambda, \kappa}$ from the conclusion of 1.4 holds.

Proof. As κ is regular apply 1.7 and then 1.4. $\square_{1.8}$

Now 1.8 was a statement I have long wanted to know, still sometimes we want to have “ $C_\delta \subseteq E$, $\text{otp}(C) = \delta(*)$ ”, $\delta(*)$ not a regular cardinal. We shall deal with such problems.

1.9 Claim. *Suppose*

- (i) $\lambda > \kappa > \aleph_0$, λ and κ are regular cardinals
- (ii) $\bar{\mathcal{P}}_\ell = \langle \mathcal{P}_{\ell, \alpha} : \alpha < \kappa \rangle$ for $\ell = 1, 2$, where $\mathcal{P}_{1, \alpha}$ is a family of $< \lambda$ closed subsets of α , $\mathcal{P}_{2, \alpha}$ is a family of $\leq \lambda$ clubs of α and $[C \in \mathcal{P}_{2, \alpha} \ \& \ \beta \in C \Rightarrow C \cap \beta \in \bigcup_{\gamma < \alpha} \mathcal{P}_{1, \gamma}]$
- (iii) $I_{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2} =: \{S \subseteq \kappa : \text{for some club } E \text{ of } \kappa \text{ for no } \delta \in S \cap E \text{ is there } C \in \mathcal{P}_{2, \alpha}, C \subseteq E\}$ is a proper ideal on κ .

Then we can find $\bar{\mathcal{P}}_\ell^* = \langle \mathcal{P}_{\ell, \alpha}^* : \alpha < \lambda \rangle$ for $\ell = 1, 2$ such that:

- (A) $\mathcal{P}_{1, \alpha}^*$ is a family of $< \lambda$ closed subsets of α
- (B) $\beta \in \text{nacc}(C) \ \& \ C \in \mathcal{P}_{1, \alpha}^* \Rightarrow C \cap \beta \in \mathcal{P}_{1, \beta}^*$
- (C) $\mathcal{P}_{2, \delta}^*$ is a family of $\leq \lambda$ clubs of δ (for δ limit $< \lambda$ such that) $[\beta \in \text{nacc}(C) \ \& \ C \in \mathcal{P}_{2, \delta}^* \Rightarrow C \cap \beta \in \mathcal{P}_{1, \beta}^*]$
- (D) for every club E of λ for some strictly increasing continuous sequence $\langle \delta_\zeta : \zeta \leq \kappa \rangle$ of ordinals $< \lambda$ we have $\{\zeta < \kappa : \zeta \text{ limit, and for some } C \in \mathcal{P}_{2, \zeta} \text{ we have: } \{\delta_\epsilon : \epsilon \in C\} \in \mathcal{P}_{2, \delta_\zeta}^* \text{ (hence } [\xi \in \text{nacc}(C) \Rightarrow \{\delta_\epsilon : \epsilon \in C \cap \xi\} \in \mathcal{P}_{1, \delta_\xi}^*]) \} \equiv \kappa \text{ mod } I_{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2}$
- (E) we have e_δ a club of δ of order type $\text{cf}(\delta)$ for any limit $\delta < \lambda$; such that for any $C \in \bigcup_{\alpha < \lambda} \mathcal{P}_{2, \alpha}^*$ for some $\delta < \lambda$, $\text{cf}(\delta) = \kappa$ and $C' \in \bigcup_{\beta < \kappa} \mathcal{P}_{2, \beta}$ we have $C = \{\gamma \in e_\delta : \text{otp}(e_\delta \cap \gamma) \in C'\}$.

Proof. Same proof as 1.5. (Note that without loss of generality $[C \in \mathcal{P}_{1,\alpha} \ \& \ \beta < \alpha < \kappa \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta}]$).

1.10 Conclusion. If $\delta(*)$ is a limit ordinal and $\lambda = \text{cf}(\lambda) > |\delta(*)|^+$ then we can find $\bar{\mathcal{P}}_\ell^* = \langle \mathcal{P}_{\ell,\alpha}^* : \alpha < \lambda \rangle$ for $\ell = 1, 2$ and stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*))\}$ such that:

- $\oplus_{\bar{\mathcal{P}}_1^*, \bar{\mathcal{P}}_2^*}^{\lambda, \delta(*)}$ (A) $\mathcal{P}_{1,\alpha}^*$ is a family of $< \lambda$ closed subsets of α each of order type $< \delta(*)$
- (B) $\beta \in \text{nacc}(C) \ \& \ C \in \mathcal{P}_{1,\alpha}^* \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta}^*$
- (C) $\mathcal{P}_{2,\delta}^*$ is a family of $\leq \lambda$ clubs of δ (yes, maybe $= \lambda$) of order type $\delta(*)$, and $[\beta \in \text{nacc}(C) \ \& \ C \in \mathcal{P}_{2,\delta}^* \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta}^*]$
- (D) for every club E of λ for some $\delta \in E \cap S$, $\text{cf}(\delta) = \text{cf}(\delta(*))$ and there is $C \in \mathcal{P}_{2,\beta}^*$ such that $C \subseteq E$.

Proof. If $\lambda = |\delta(*)|^{++}$ (or any successor of regulars) use [Sh:e, ChIII,6.4](2) or [Sh 365, 2.14](2)((c)+(d)).

If $\lambda > |\delta(*)|^{++}$ let $\kappa = |\delta(*)|^{++}$ and let $S_1 = \{\delta < \kappa^{++} : \text{cf}(\delta) = \text{cf}(\delta(*))\}$; applying the previous sentence we get $\bar{\mathcal{P}}_1^*, \bar{\mathcal{P}}_2^*$ satisfying $\oplus_{\bar{\mathcal{P}}_1^*, \bar{\mathcal{P}}_2^*, S_1}^{\kappa^{++}, \delta(*)}$, hence satisfying the assumption of 1.9 so we can apply 1.9. $\square_{1.10}$

1.11 Definition. $+\oplus_{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_{2,S}}^{\lambda, \delta(*)}$ is defined as in 1.10 except that we replace (C) by

(C)⁺ $\mathcal{P}_{2,\delta}^*$ is a family of $< \lambda$ clubs of δ of order type $\delta(*)$.

1.12 Remark. Note that if $\mathcal{P}_\alpha = \mathcal{P}_{1,\alpha} \cup \mathcal{P}_{2,\alpha}$, $|\mathcal{P}_{2,\alpha}| \leq 1$, $\mathcal{P}_{1,\alpha} = \{C \in \mathcal{P}_\alpha : \text{otp}(C) < \delta(*)\}$, $\mathcal{P}_{2,\alpha} = \{C \in \mathcal{P}_\alpha : \text{otp}(C) = \delta(*)\}$ then $+\oplus_{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_{2,S}}^{\lambda, \delta(*)} \Leftrightarrow \oplus_{\bar{\mathcal{P}}_S}^{\lambda, \delta(*)}$ mod.

1.13 Claim. Suppose $\lambda = \text{cf}(\lambda) > |\delta(*)|^+$, $\delta(*)$ a limit ordinal, additively indecomposable (i.e. $\alpha < \delta(*) \Rightarrow \alpha + \alpha < \delta(*)$), $\oplus_{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_{2,S}}^{\lambda, \delta(*)}$ from 1.10 and

(*) $\alpha \in S \Rightarrow |\mathcal{P}_{2,\alpha}| \leq |\alpha|$.

(Note: a non-stationary subset of S does not count; e.g. for λ successor cardinal the α with $|\alpha|^+ < \lambda$. Note: ${}^+\oplus_{\mathcal{P}_1, \mathcal{P}_{2,S}}^{\lambda, \delta(*)}$ holds by $(*)$ and if λ is successor then ${}^+\oplus_{\mathcal{P}_1, \mathcal{P}_{2,S}}^{\lambda, \delta(*)}$ suffice).

Then for some stationary $S_1 \subseteq S$ and $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ we have: $\mathcal{P}_\alpha \subseteq \mathcal{P}_{1,\alpha} \cup \mathcal{P}_{2,\alpha}$ and:

- $*\otimes_{\bar{\mathcal{P}}, S_1}^{\lambda, \delta(*)}$ (i) \mathcal{P}_α is a family of closed subsets of α , $|\mathcal{P}_\alpha| < \lambda$
- (ii) $\text{otp} C < \delta(*)$ if $C \in \mathcal{P}_\alpha, \alpha \notin S_1$
- (iii) if $\alpha \in S_1$ then: $\mathcal{P}_\alpha = \{C_\alpha\}$, $\text{otp}(C_\alpha) = \delta(*)$,
 C_α a club of α disjoint to S_1
- (iv) $C \in \mathcal{P}_\alpha$ & $\beta \in \text{nacc}(C) \Rightarrow \beta \cap C \in \mathcal{P}_\beta$
- (v) for any club E of λ for some $\delta \in S_1$ we have $C_\delta \subseteq E$.

1.14 Remark. Note there are two points we gain: for $\alpha \in S_1$, \mathcal{P}_α is a singleton (similarly to 1.4 where we have $(\exists \leq^1 C \in \mathcal{P}_\delta)[\text{otp}(C) = \delta(*)]$), and an ordinal α cannot have a double role $-C_\alpha$ a guess (i.e. $\alpha \in S_1$) and C_α is a proper initial segment of such C_δ . When $\delta(*)$ is a regular cardinal this is easier.

Proof. Let $\mathcal{P}_{2,\alpha} = \{C_{\alpha,i} : i < \alpha\}$ (such a list exists as we have assumed $|\mathcal{P}_{2,\alpha}| \leq |\alpha|$, we ignore the case $\mathcal{P}_{2,\alpha} = \emptyset$). Now

- $(*)_0$ for some $i < \lambda$ for every club E of λ for some $\delta \in S \cap E$ we have $C_{\delta,i} \setminus E$ is bounded in α
 [Why? If not, for every $i < \lambda$ there is a club E_i of λ such that for no $\delta \in S \cap E$ is $C_{\delta,i} \setminus E$ bounded in α . Let $E^* = \{j < \lambda : j \text{ a limit ordinal, } j \in \bigcap_{i < j} E_i\}$, it is a club of λ , hence for some $\delta \in S \cap E^*$ and $C \in \mathcal{P}_{2,\delta}$ we have $C \subseteq E^*$. So for some $i < \alpha$, $C = C_{\delta,i}$, so $C \subseteq E^* \subseteq E_i \cup i$ hence $C_{\delta,i} \setminus i \subseteq E_i$, contradicting the choice of E_i .]
- $(*)_1$ for some $i < \lambda$ and $\gamma < \delta(*)$, letting $C_\delta =: C_{\delta,i} \setminus \{\zeta \in C_{\delta,i} : \text{otp}(\zeta \cap C_{\delta,i}) < \gamma\}$ we have: for every club E of λ for some $\delta \in S \cap E$ we have: $C_\delta \subseteq E$
 [Why? Let $i(*)$ be as in $(*)_0$, and for each $\gamma < \delta(*)$ suppose E_γ exemplify the failure of $(*)_1$ for $i(*)$ and γ , now $\bigcap_{\gamma < \delta(*)} E_\gamma$ is a club of λ exemplifying the failure of $(*)_0$ for $i(*)$ contradiction. So for some $\gamma < \delta(*)$ we succeed.]
- $(*)_2$ Without loss of generality $|\mathcal{P}_{2,\alpha}| \leq 1$, so let $\mathcal{P}_{2,\alpha} = \{C_\alpha\}$
 [Why? Let i, γ and C_δ (for $\delta \in S$) be as in $(*)_1$ and use $\mathcal{P}'_{1,\alpha} = \{C \setminus \{\zeta \in C : \text{otp}(\zeta \cap C) < \gamma\} : C \in \mathcal{P}_{1,\alpha}\}$, $\mathcal{P}'_{2,i} = \{C_\delta\}$.]

- (*)₃ for some $h : \lambda \rightarrow |\delta(*)|^+$, for every $\alpha \in S$ we have $h(\alpha) \notin \{h(\beta) : \beta \in C_\alpha\}$
 [Why? Choose $h(\alpha)$ by induction on α .]
 (*)₄ for some $\beta < |\delta(*)|^+$ for every club E of λ , for some $\delta \in S \cap h^{-1}(\{\beta\})$, $C_\delta \subseteq E$
 [Why? If for each β there is a counterexample E_β then $\cap\{E_\beta : \beta < |\delta(*)|^+\}$ is a counterexample for (*)₂.]

Now we have gotten the desired conclusion. □_{1.13}

1.15 Claim. *If $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$, $S \in I[\lambda]$, $\kappa^+ < \lambda = \text{cf}(\lambda)$, then for some stationary $S_1 \subseteq S$ and $\bar{\mathcal{P}}_1$ we have $^*\oplus_{\bar{\mathcal{P}}_1, S_1}^{\lambda, \delta(*)}$.*

Proof. Same proof as 1.4 (plus (*)₃, (*)₄ in the proof of 1.10). □_{1.15}

1.16 Claim. *Assume $\lambda = \mu^+$, $|\delta(*)| < \mu$ and $\text{cf}(\delta(*)) \neq \text{cf}(\mu)$.*

Then we can find stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta)()\}$ and $\bar{\mathcal{P}}$ such that $^*\otimes_{\bar{\mathcal{P}}, S}^{\lambda, \delta(*)}$.*

Remark. This strengthens 1.10.

Proof. Case (α). μ regular.

By [Sh:e, Ch.III,6.4](2), [Sh 365, 2.14](2)((c)+(d)).

Case β. μ singular.

Let $\theta =: \text{cf}(\mu)$, $\sigma =: |\delta(*)|^+ + \theta^+$ and $\mu = \sum_{\zeta < \theta} \mu_\zeta$, $\langle \mu_\zeta : \zeta < \theta \rangle$ strictly increasing,

$\mu_0 > \sigma$ and for each $\alpha < \lambda$ let $\alpha = \bigcup_{\zeta < \theta} A_{\alpha, \zeta}$, $\langle A_{\alpha, \zeta} : \zeta < \theta \rangle$ increasing, $|A_{\alpha, \zeta}| \leq \mu_\zeta$.

By 1.8 there is a sequence $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ and stationary $S_1 \subseteq \{\delta < \lambda : \text{cf}(\delta) = \sigma\}$ such that $\oplus_{\bar{\mathcal{P}}, S_1}^{\lambda, \sigma}$ of 1.4 holds. Let $\cup \{\mathcal{P}_\alpha : \alpha < \lambda\} \cup \{\emptyset\}$ be $\{C_\alpha : \alpha < \lambda\}$ such that $C_\alpha \subseteq \alpha$, $[\alpha \in S_1 \Rightarrow C_\alpha \in \mathcal{P}_\alpha \ \& \ \text{otp}(C_\alpha) = \sigma]$ and $[\alpha \notin S_1 \Rightarrow \text{otp}(C_\alpha) < \sigma]$. For some club E_1^* of λ , $[\alpha \in E_1^* \Rightarrow \bigcup_{\beta < \alpha} \mathcal{P}_\beta = \{C_\beta : \beta < \alpha\}]$.

Looking again at $\oplus_{\bar{\mathcal{P}}, S_1}^{\lambda, \sigma}$, we can assume $S_1 \subseteq E_1^*$ & $(\forall \delta)[\delta \in S_1 \Rightarrow C_\delta \subseteq E_1^*]$, hence

(*) $\delta \in S_1 \ \& \ \alpha \in \text{nacc } C_\delta \Rightarrow \alpha \cap C_\delta \in \{C_\beta : \beta < \text{Min}(C_\delta \setminus (\alpha + 1))\}$.

So as we can replace every C_α by $\{\beta \in C_\alpha : \text{otp}(C_\alpha \cap \beta)\}$ is even, without loss of generality [because we can replace every C_α by $\{\beta \in C_\alpha : \text{otp}(\beta \cap C_\alpha) \text{ is even}\}$, without loss of generality (check)]

$$(*)^+ \delta \in S_1 \ \& \ \alpha \in \text{nacc } C_\delta \Rightarrow \alpha \cap C_\delta \in \{C_\beta : \beta < \alpha\}.$$

Without loss of generality $[\beta \in A_{\alpha,\zeta} \Rightarrow C_\beta \subseteq A_{\alpha,\zeta}]$ (just note $|C_\beta| \leq \sigma < \mu_\zeta$) and $\alpha \in A_{\beta,\zeta} \Rightarrow A_{\alpha,\zeta} \subseteq A_{\beta,\zeta}$. For $\alpha \in S_1$ let $C_\alpha = \{\beta_{\alpha,\epsilon} : \epsilon < \sigma\}$ ($\beta_{\alpha,\epsilon}$ increasing in ϵ) and let $\beta_{\alpha,\epsilon}^* \in [\beta_{\alpha,\epsilon}, \beta_{\alpha,\epsilon+1})$ be minimal such that $C_\alpha \cap \beta_{\alpha,\epsilon+1} = C_{\beta_{\alpha,\epsilon}^*}$ (exists as $\delta \in S_1 \Rightarrow C_\delta \subseteq E_1^*$). Without loss of generality every C_α is an initial segment of some C_β , $\beta \in S_1$ (if not, we redefine it as \emptyset).

- (*)₁ there are $\gamma = \gamma(*) < \theta$ and stationary $S_2 \subseteq S_1$ such that for every club E of λ , for some $\delta \in S_2$ we have: $C_\delta \subseteq E$, and for arbitrarily large $\epsilon < \sigma$, $\beta_{\delta,\epsilon}^* \in A_{\beta_{\delta,\epsilon+1},\gamma}$.
 [Why? If not, for every $\gamma < \theta$ (by trying $\gamma(*) = \gamma$) there is a club E_γ of λ exemplifying the failure of (*)₁ for γ . Let $E = \bigcap_{\gamma < \theta} E_\gamma \cap E_1^*$, so E is a club of λ , hence

$$S' =: \{\delta : \delta < \lambda, \delta \in S_1 (\text{so } \text{cf}(\delta) = \sigma) \text{ and } C_\delta \subseteq E\}$$

is a stationary subset of λ . For each $\delta \in S'$ and $\epsilon < \sigma$ for some $\gamma = \gamma(\delta, \epsilon) < \theta$ we have $\beta_{\delta,\epsilon}^* \in A_{\beta_{\delta,\epsilon+1},\gamma}$, but as $\sigma = \text{cf}(\sigma) \neq \text{cf}(\theta) = \theta$ for some $\gamma(\delta)$, $\{\epsilon < \sigma : \epsilon\gamma(\delta, \epsilon) = \gamma(\delta)\}$ is unbounded in σ . But $\delta \in E_{\gamma(\delta)}$, contradiction.]

- (*)₂ Without loss of generality: if $\beta \in \text{nacc}(C_\alpha)$, $\alpha < \lambda$ then $(\exists \xi \in A_{\beta,\gamma(*)})[\beta > \xi > \sup(\beta \cap C_\alpha) \ \& \ \beta \cap C_\alpha = C_\xi]$.
 [Why? Define C'_α for $\alpha < \lambda$:
 $C_\alpha^0 = \{\beta : \beta \in \text{nacc}(C_\alpha) \text{ and } (\exists \xi \in A_{\beta,\gamma(*)})[\beta > \xi \geq \sup(\beta \cap C_\alpha) \ \& \ \beta \cap C_\alpha = C_\xi]\}$.
 C'_α is: \emptyset if $\alpha \in S_2$, $\alpha > \sup(C_\alpha^0)$
 $\alpha \cap \text{closure of } C_\alpha^0$ otherwise.] Now $\langle C_\alpha : \alpha < \lambda \rangle$ can be replaced by $\langle C'_\alpha : \alpha < \lambda \rangle$.]

- (*)₃ For some $\gamma_1 = \gamma_1(*) < \theta$ for every club E of λ for some $\delta \in E : \text{cf}(\delta) = \text{cf}(\delta(*))$, and there is a club e of δ satisfying: $e \subseteq E$, $\text{otp}(e)$ is $\delta(*)$, and for arbitrarily large $\beta \in \text{nacc}(e)$ we have $e \cap \beta \in \{C_\zeta : \zeta \in A_{\delta,\gamma_1}\}$.
 [Why? If not, for each $\gamma_1 < \theta$ there is a club E_{γ_1} of λ for which there is no δ as required. Let $E =: \bigcap_{\gamma_1 < \theta} E_{\gamma_1}$, so E is a club of λ hence for some $\alpha \in \text{acc}(E) \cap S_2$, $C_\alpha \subseteq E$. Letting again $C_\alpha = \{\beta_{\alpha,\epsilon} : \epsilon < \sigma\}$ (increasing), $C_\alpha \cap \beta_{\alpha,\epsilon} = C_{\delta,\beta_{\delta,\epsilon}^*}$ where $\beta_{\delta,\epsilon}^* \in A_{\beta_{\delta,\epsilon+1},\gamma(*)}$ clearly $\delta =: \beta_{\alpha,\delta(*)}$, $e = \{\beta_{\delta,\epsilon} :$

$\epsilon < \delta(*)$ satisfies the requirements except the last. As $\text{cf}(\delta(*)) \neq \text{cf}(\mu)$, for some $\gamma_1(*) < \theta$, $\gamma_1(*) \geq \gamma(*)$ and $\{\epsilon < \delta(*) : \beta_{\delta, \epsilon}^* \in A_{\beta_{\delta, \delta(*), \gamma_1(*)}}\}$ is unbounded in $\delta(*)$. Clearly $\delta =: \beta_{\alpha, \delta(*)}$, $e =: C_\alpha \cap \delta$ satisfies the requirement. Now this contradicts the choice of $E_{\gamma_1(*)}$.]

(*)₄ For some club E^a of λ , for every club $E^b \subseteq E^a$ of λ , for some $\delta \in E^b$ we have:

- (a) $\text{cf}(\delta) = \text{cf}(\delta(*))$
- (b) for some club e of $\delta : e \subseteq E^b$, $\text{otp}(e) = \delta(*)$, and for arbitrarily large $\beta \in \text{nacc}(e)$ we have $e \cap \beta \in \{C_\xi : \xi \in A_{\delta, \gamma_1(*)}\}$
- (c) for every $\beta \in A_{\delta, \gamma_1(*)}$ we have: $C_\beta \subseteq E^a \Rightarrow C_\beta \subseteq E^b$ (we could have demanded $C_\beta \cap E^a = C_\beta \cap E^b$).
[Why? If not we choose E_i for $i < \mu_{\gamma_1(*)}^+$ by induction on i , $[j < i \Rightarrow E_i \subseteq E_j]$, E_i a club of λ , and E_{i+1} exemplify the failure of E_i as a candidate for E^a . So $\bigcap_i E_i$ is a club of λ hence by (*)₃ there are δ and e as there. Now $\langle \{\beta \in A_{\delta, \gamma_1(*)} : C_\beta \subseteq E_i\} : i < \mu_{\gamma_1(*)}^+ \rangle$ is a decreasing sequence of subsets of $A_{\delta, \gamma_1(*)}$ of length $\mu_{\gamma_1(*)}^+$, and $|A_{\delta, \gamma_1(*)}| \leq \mu_{\gamma_1(*)}$, hence it is eventually constant. So for every i large enough, δ contradicts the choice of E_{i+1} .]

* * *

Let $S = \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*)), \text{ and there is a club } e = e_\delta \text{ of } \delta \text{ satisfying: } e \subseteq E^a, \text{otp}(e) = \delta(*), \alpha \in \text{nacc}(e) \Rightarrow e \cap \alpha \in A_{\alpha, \gamma(*)} \text{ and for arbitrarily large } \beta \in \text{nacc}(e) \text{ we have } e \cap \beta \in \{C_\xi : \xi \in A_{\delta, \gamma(*)}\}\}$.

So S is stationary, let for $\delta \in S$, C_δ^* be an e as above. For $\alpha < \lambda$ let $\mathcal{P}_{1, \alpha} = \{C_\beta : \beta \leq \alpha, \beta \in A_{\alpha, \gamma_2(*)}\}$

- (*)₅(a) for every club E of λ , for some $\delta \in S$, $C_\delta^* \subseteq E$
- (b) C_δ^* is a club of δ , $\text{otp}(C_\delta^*) = \delta(*)$
- (c) if $\beta \in \text{nacc } C_\delta^* (\delta \in S)$ then $C_\delta^* \cap \beta \in \mathcal{P}_{1, \beta}$
- (d) $|\mathcal{P}_{1, \beta}| \leq \mu_{\gamma(*)}$, $\mathcal{P}_{1, \beta}$ is a family of closed subsets of β of order type $< \delta(*)$,
[Why? This is what we have proved in (*)₄; noting that in (*)₄ in (b), (e) is not uniquely determined, but by (c) every “reasonable” candidate is O.K.]

Now repeating (*)₃, (*)₄ of the proof of 1.13, and we finish. □_{1.16}

1.17 Claim. 1) Assume $\lambda = \mu^+$, $|\delta(*)| < \mu$, $\aleph_0 < \text{cf}(\delta(*)) = \text{cf}(\mu)(< \mu)$; then we can find stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\delta(*))\}$ and $\bar{\mathcal{P}}$ such that $* \otimes_{\bar{\mathcal{P}}, S}^{\lambda, \delta(*)}$, except when:

\oplus for every regular $\sigma < \mu$, we can find $h : \sigma \rightarrow \text{cf}(\mu)$ such that for no δ, ϵ do we have: if $\delta < \sigma$, $\text{cf}(\delta) = \text{cf}(\mu)$, $\epsilon < \text{cf}(\mu)$ then $\{\alpha < \delta : h(\alpha) < \epsilon\}$ is not a stationary subset of δ .

2) In 1.16 and 1.17(1) we can have $\mu > \sup\{|\mathcal{P}_\alpha| : \alpha < \lambda\}$.

3) If 1.17(2) if μ is strong limit we can have $|\mathcal{P}_\alpha| \leq 1$ for each α .

Remark. Compare with [Sh 186, §3].

Proof. Left to the reader (reread the proof of 1.16 and [Sh 186, §3]).

1.18 Claim. 1) Let κ be regular uncountable and we have global choice (or restrict ourselves to $\lambda < \lambda^*$). We can choose for each regular $\lambda > \kappa^+$, $\bar{\mathcal{P}}^\lambda = \langle \mathcal{P}_\alpha^\lambda : \alpha < \lambda \rangle$ (assuming global choice) such that:

- (a) for each λ , $\mathcal{P}_\alpha^\lambda$ is a family of $\leq \lambda$ of closed subsets of α of order type $< \kappa$.
- (b) if χ is regular, F is the function $\lambda \mapsto \bar{\mathcal{P}}^\lambda$ (for λ regular $< \chi$), $\aleph_0 < \kappa = \text{cf}(\kappa)$, $\kappa^{++} < \chi$, $x \in \mathcal{H}(\chi)$ then we can find $\bar{N} = \langle N_i : i \leq \kappa \rangle$, an increasing continuous chain of elementary submodels of $(\mathcal{H}(\chi), \in, <_\chi^*, F)$, $\langle N_j : j \leq i \rangle \in N_{i+1}$, $\|N_i\| = \aleph_0 + |i|$, $x \in N_0$ such that:
 - (*) if $\kappa^+ < \theta = \text{cf}(\theta) \in N_i$, then for some club C of $\text{sup}(N_\kappa \cap \theta)$ of order type κ ; for any $j_1^i < j < \kappa$ we have:

$$C \cap \text{sup}(N_j \cap \theta) \in N_{j+1}, \text{otp}(C \cap \text{sup}(N_j \cap \theta)) = j.$$

2) We can above have $|\mathcal{P}_\alpha^\lambda| < \lambda$.

Proof. 1) Let $\langle C_\alpha : \alpha \in S \rangle$ be such that $S \subseteq \{\alpha \leq \kappa^{++} : \text{cf}(\alpha) \leq \kappa\}$ is stationary, $\text{otp}(C_\alpha) \leq \kappa$, $[\beta \in C_\alpha \Rightarrow C_\beta = \beta \cap C_\alpha]$, C_α a closed subset of α , $[\alpha \text{ limit} \Rightarrow \alpha = \text{sup}(C_\alpha)]$, $\{\alpha \in S : \text{cf}(\alpha) = \kappa\}$ stationary, and for every club E of κ^{++} there is $\delta \in S$, $\text{cf}(\delta) = \kappa$, $C_\delta \subseteq E$. For $i \in \kappa^{++} \setminus S$ let $C_i = \emptyset$. Now for every regular $\lambda > \kappa^+$ and $\alpha \leq \lambda$, let $e_\alpha^\lambda \subseteq \alpha$ be a club of α of order type $\text{cf}(\alpha)$. For λ as above and for $\alpha \leq \lambda$ limit let $\bar{\mathcal{P}}_\alpha^\lambda = \{\{i \in e_\delta : i < \alpha, \text{otp}(e_\delta \cap i) \in C_\beta\} : \delta < \lambda \text{ has cofinality } \kappa^{++}, \text{ and } \beta \in S\}$. Given $x \in H(\chi)$, we choose by induction on $i < \kappa^{++}$, M_i , N_i such that:

$$N_i \prec M_i \prec (\mathcal{H}(\chi), \in, <_\chi^*, F)$$

$$\|M_i\| = |i| + \aleph_0$$

$$\|N_i\| = |C_i| + \aleph_0$$

$M_i(i < \kappa^{++})$ is increasing continuous

$$x \in M_0,$$

$$\langle M_j : j \leq i \rangle \in M_{i+1}$$

N_i is the Skolem Hull of $\{\langle N_j : j \in C_\zeta \rangle : \zeta \in C_i\}$.

We leave the checking to the reader.

2) We imitate the proof of 1.5.

□_{1.18}

§2 MEASURING $[\lambda]^{<\kappa}$

We prove here that two natural ways to measure $\mathcal{S}_{<\kappa}(\lambda)$ for κ regular uncountable, give the same cardinal: the minimal cardinality of a cofinal subset; i.e. its cofinality (i.e. $\text{cov}(\lambda, \kappa, \kappa, 2)$) and the minimal cardinality of a stationary subset. The theorem is really somewhat stronger: for appropriate normal ideal on $\mathcal{S}_{<\kappa}(\lambda)$, some member of the dual filter has the right cardinality.

The problem is natural and I did not trace its origin, but until recent years it seems (at least to me) it surely is independent, and find it gratifying we get a clean answer. I thank P. Matet and M. Gitik of reminding me of the problem.

We then find applications to Δ -systems and largeness of $\check{I}[\lambda]$.

2.1 Definition. 1) Let $(\bar{C}, \bar{\mathcal{P}}, Z) \in \mathcal{T}^*[\theta, \kappa]$ when:

- (i) $\aleph_0 < \kappa = \text{cf}(\kappa) < \theta = \text{cf}(\theta)$,
- (ii) $\bar{C} = \langle C_\delta : \delta \in S \rangle$, $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S \rangle$, $Z = \langle \langle \mathcal{P}_\delta : \delta \in S \rangle$
- (iii) $S \subseteq \theta$, S is stationary (we shall write $S = S(\bar{C})$),
- (iv) C_δ is an unbounded subset of δ , (not necessarily closed)
- (v) $\text{id}^a(\bar{C})$ is a proper ideal (i.e. for every club E of θ for some $\delta \in S$, $C_\delta \subseteq E$)
- (vi) $\bigwedge_{\delta \in S} \text{otp}(C_\delta) < \kappa$, (hence $[\delta \in S \Rightarrow \text{cf}(\delta) < \kappa]$)
- (vii) (α) \mathcal{P}_δ is a family of bounded subsets of C_δ , directed by the partial order $<_{\mathcal{P}_\delta}$ which is a partial order on $\mathcal{P}^* = \{x \cap \alpha : x \in \mathcal{P}_\delta \text{ for some } \delta \in S \text{ and } \alpha < \theta\}$ satisfying $y <_{\mathcal{P}_\delta} z \Rightarrow y \subseteq z$, (but see parts (1A),(1B))
- (β) $\bigcup_{x \in \mathcal{P}_\delta} x = C_\delta$, and $|\mathcal{P}_\delta| < \kappa$
- (viii) for some¹ list $\langle b_i^* : i < \theta \rangle$ of $\bigcup_{\alpha \in S} \mathcal{P}_\alpha \cup \{\emptyset\}$ satisfying $b_i^* \subseteq i$ we have: for every $\alpha \in S$ we have $\mathcal{P}_\alpha \subseteq \{b_j^* : j < \alpha\}$
- (ix) for $x \in \bigcup_{\delta \in S} \mathcal{P}_\delta$ we have the set $\mathcal{P}_x := \{y \in \bigcup_{\delta \in S} \mathcal{P}_\delta : y <_{\mathcal{P}_\delta} x\}$ has cardinality κ .

¹a sufficient condition is:

(viii)⁺ for every $\alpha < \theta$ the set $\mathcal{P}_\alpha^* =: \{a \cap \alpha : \text{for some } \delta \in S \text{ we have } \alpha < \delta \in S, a \in \mathcal{P}_\delta \text{ and } \alpha \in C_\delta\}$ has cardinality $< \theta$ or at least

1A) If each $<_{\mathcal{P}_\delta}$ is inclusion we may omit it.

1B) If $<_*$ is a partial order of $\bigcup_{\delta \in S} \mathcal{P}_\delta$ and $\delta \in S \Rightarrow <_{\mathcal{P}_\delta} = <_* \upharpoonright \mathcal{P}_\delta$ then we may write $<_*$ instead of Z .

2) $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$, if $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ where $\delta \in S(\bar{C}) \Rightarrow \bar{\mathcal{P}}_\delta = \{C_\delta \cap \alpha : \alpha \in C_\delta\}$.

3) $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$ if $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ where $\delta \in S(\bar{C}) \Rightarrow \bar{\mathcal{P}}_\delta = [C_\delta]^{<\aleph_0}$.

Note that:

2.2 Claim. 1) If $\theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \sigma = \text{cf}(\sigma)$, then there is $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$ such that:

$$\{\delta \in S(\bar{C}) : \text{cf}(\delta) = \sigma\} \neq \emptyset \bmod \text{id}^a(\bar{C}).$$

2) If $S \subseteq \{\delta < \theta : \text{cf}(\delta) < \kappa\}$ is stationary, \bar{C} an S -club system, $|C_\delta| < \kappa$, and $\text{id}^a(\bar{C})$ a proper ideal, then $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$.

3) In (2) if in addition for each $\alpha < \theta$ we have $|\{C_\delta \cap \alpha : \alpha \in C_\delta, \delta \in S\}| < \theta$ then $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$.

4) If θ is a successor of regular then in part (2) we can demand $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$ each C_δ closed.

5) If $\theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \sigma = \text{cf}(\sigma)$, then there is $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$ such that: $\{\delta \in S(\bar{C}) : \text{cf}(\delta) = \sigma\} \neq \emptyset \bmod \text{id}^a(\bar{C})$.

6) If $\theta = \text{cf}(\theta) = \text{cf}(\kappa) > \sigma = \text{cf}(\sigma)$ and $S \in \check{I}[\theta]$ is stationary then there is $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$ such that $S(\bar{C}) = S$.

Proof. 1) Let $S_0 \subseteq \{\delta < \theta : \text{cf}(\delta) = \sigma\}$ be stationary, C_δ^0 a club of δ of order type σ . By [Sh 365, §2], for some club E of λ letting $S = S_0 \cap \text{acc}(E)$ and letting, for $\delta \in S$, $C_\delta = g\ell(C_\delta^0, E) = \{\sup(\alpha \cap E) : \alpha \in C_\delta^0\}$ we have $S \notin \text{id}^a(\langle C_\delta : \delta \in S_0 \rangle)$, now use part (2).

2) Check.

3) Check.

4) By [Sh 351, §4], [Sh:e, Ch.IV, 3.4](2) or [Sh 365, 2.14](2)((c)+(d)) but see [Sh:E12].

5) By 1.7 and 1.15 (so we use the non-accumulation points).

6) Similarly. □_{2.2}

Remember (see [Sh 52, §3]).

2.3 Definition. 1) $\mathcal{D}_{<\kappa}^\kappa(\lambda)$ is the filter on $[\lambda]^{<\kappa}$ defined by:

for $X \subseteq [\lambda]^{<\kappa}$:

$X \in \mathcal{D}_{<\kappa}^\kappa(\lambda)$ iff there is a function F with domain the set of sequences of length $< \kappa$ with elements from $[\lambda]^{<\kappa}$ and F is into $[\lambda]^{<\kappa}$ such that: if $a_\zeta \in [\lambda]^{<\kappa}$ for $\zeta < \kappa$, is \subseteq -increasing continuous and for each $\zeta < \kappa$ we have $F(\langle \dots, a_\xi, \dots \rangle)_{\xi \leq \zeta} \subseteq a_{\zeta+1}$ then $\{\zeta < \kappa : a_\zeta \in X\} \in \mathcal{D}_\kappa$.

(recall that \mathcal{D}_κ the filter generated by the family of clubs of κ).

Similarly

2.4 Definition. For $\lambda \geq \theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0$, $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ and set X of cardinality $\geq \kappa$ we define a filter $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda)$ on $[\lambda]^{<\kappa}$; (letting, e.g. $\chi = \beth_{\omega+1}(\lambda)$):

$Y \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda)$ iff $Y \subseteq [\lambda]^{<\kappa}$ and for some $\mathbf{x} \in \mathcal{H}(\chi)$, for every $\langle N_\alpha, N_a^* : \alpha < \theta, a \in \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle$ satisfying \otimes below, also there is $A \in \text{id}^a(\bar{C})$ such that: $\delta \in S(\bar{C}) \setminus A \Rightarrow \bigcup_{a \in \mathcal{P}_\delta} N_a^* \cap \lambda \in Y$ where, letting $\mathcal{P} = \bigcup \{\mathcal{P}_\delta : \delta \in S\}$,

$\otimes(i)$ $N_\alpha \prec (\mathcal{H}(\chi), \in, <_\chi^*)$

(ii) $\|N_\alpha\| < \theta$,

(iii) $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$

(iv) $\langle N_\alpha : \alpha < \theta \rangle$ is increasing continuous

(v) $N_a^* \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ for $a \in \bigcup_{\delta \in S} \mathcal{P}_\delta$

(vi) $\|N_a^*\| < \kappa$, $N_a^* \cap \kappa$ an initial segment of κ

(vii) $b \subseteq a$ (both in $\bigcup_{\delta \in S} \mathcal{P}_\delta$) implies $N_b^* \prec N_a^*$

(viii) if $\alpha \in a \in \bigcup_{\delta \in S} \mathcal{P}_\delta$ then $\langle N_\beta, N_b^* : \beta \leq \alpha, b \subseteq a, b \in \{b_i^* : i \leq \alpha\} \subseteq \mathcal{P} = \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle$ belongs to N_a^*

(ix) $\langle N_\beta, N_b^* : \beta \leq \alpha, b \subseteq \alpha + 1, b \in \{b_i^* : i \leq \alpha + 1\} \subseteq \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle$ belongs to $N_{\alpha+1}$

(x) $a \subseteq N_a^*$ and $\alpha \in a \Rightarrow \alpha \cap a \in N_a^*$

(xi) $a \subseteq \alpha, a \in \mathcal{P}$ implies $N_a^* \in N_{\alpha+1}$ (follows from (ix) by clause (viii) of Definition 2.1(1))

(xii) $a \in \mathcal{P}_\delta$ & $\delta \in S$ & $\alpha < \theta \Rightarrow \mathbf{x} \in N_a^*$ & $\mathbf{x} \in N_\alpha$.

Clearly

2.5 Claim. 1) Any χ that is $\mathcal{H}(\chi)$ can serve, and $\mathbf{x} = (Y, \lambda, \bar{C}, \bar{\mathcal{P}})$ is enough.
 2) $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda)$ is a (non-trivial) fine $(< \kappa)$ -complete filter on $[\lambda]^{<\kappa}$ when $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$, $\lambda \geq \theta$, hence it extends $\mathcal{D}_{<\kappa}(\lambda)$. (Remember $\text{id}^a(\bar{C})$ is a proper ideal).

Proof. Should be clear. □_{2.5}

2.6 Theorem. Suppose $\lambda > \theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0$ and $\theta = \kappa^+$. Then the following four cardinals are equal for any $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$, recalling there are such $(\bar{C}, \bar{\mathcal{P}})$ by 2.2:

$$\mu(0) = \text{cf}([\lambda]^{<\kappa}, \subseteq)$$

$$\mu(1) = \text{cov}(\lambda, \kappa, \kappa, 2) = \text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{<\kappa}, \text{ and for every } a \subseteq \lambda, |a| < \kappa \text{ there is } b \in \mathcal{P} \text{ satisfying } a \subseteq b\}$$

$$\mu(2) = \text{Min}\{|S| : S \subseteq [\lambda]^{<\kappa} \text{ is stationary}\}$$

$$\mu(3) = \mu_{(\bar{C}, \bar{\mathcal{P}})} = \text{Min}\{|Y| : Y \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda)\}.$$

2.7 Remark. 0) We thank M. Shioya for asking for a correction of an inaccuracy in the proof in a meeting in the summer of 1999 in which we answer him; this and other minor changes are done here. I thank P. Komjath for helpful comments and S. Garti for help in proofreading.

1) It is well known that if $\lambda > 2^{<\kappa}$ then the equality holds as they are all equal to $\lambda^{<\kappa}$.

2) This is close to “strong covering”.

3) Note that only $\mu(3)$ has $(\bar{C}, \bar{\mathcal{P}})$ in its definition, so actually $\mu(3)$ does not depend on $(\bar{C}, \bar{\mathcal{P}})$, recalling that by Claim 2.2 we know that $\mathcal{T}^*[\theta, \kappa]$ is not empty.

4) $\mu(0), \mu(1)$ are equal trivially.

2.8 Remark. 0) We can concentrate on the case $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^1[\theta, \kappa]$ or $\mathcal{T}^0[\theta, \kappa]$. This somewhat simplifies and is enough.

1) We can weaken in Definition 2.1(1) demand (ix) as follows:

(ix)' there is a sequence $\langle a_i, \mathcal{P}_i^* : i < \lambda \rangle$ such that

- (a) $|a_i| < \kappa$, \mathcal{P}_i^* is a family of $< \kappa$ subsets of a_i
- (b) for every $\delta \in S$ and $x \in \mathcal{P}_\delta$ for some $i < \delta$, $a_i = x$ and $(\forall b)[b \in \mathcal{P}_\delta \ \& \ b \subseteq a \Rightarrow b \in \mathcal{P}_i^*]$.

In this case 2.6, 2.7(4) (and 2.5) remains true and we can strengthen 2.2.

2) We can even use \mathcal{P}_δ with another order (not \subseteq).

Proof. Clearly $\lambda \leq \mu(0) = \mu(1) \leq \mu(2) \leq \mu(3)$ (the last — by 2.5(2)). So we shall finish by proving $\mu(3) \leq \mu(1)$, and let \mathcal{Q} exemplify $\mu(1) = \text{cov}(\lambda, \kappa, \kappa, 2)$. Let $S = S(\bar{C})$, etc.

Let χ be e.g. $\beth_3(\lambda)^+$ and let M_λ^* be the model with universe $\lambda + 1$ and all functions definable in $(\mathcal{H}(\chi), \in, <_\chi^*, \lambda, \kappa, \mu(1))$. Let M^* be an elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi^*)$ of cardinality $\mu(1)$ such that $\mathcal{Q} \in M^*$, $M_\lambda^* \in M^*$, $(\bar{C}, \bar{\mathcal{P}}) \in M^*$ and $\mu(1) + 1 \subseteq M^*$ hence $\mathcal{Q} \subseteq M^*$. It is enough to prove that $M^* \cap [\lambda]^{<\kappa}$ belongs to $\mathcal{D}_{(\bar{C}, \bar{P})}(\lambda)$.

So let N_i (for $i < \theta$), N_x^* (for $x \in \bigcup_{\delta \in S} \mathcal{P}_\delta$) be such that: they satisfy \otimes of Definition 2.4 for $\mathbf{x} := \langle M_\lambda^*, M^*, \mathcal{P}, \lambda, \kappa, (\bar{C}, \bar{\mathcal{P}}) \rangle$ so it belongs to every N_α , N_x^* . It is enough to prove that $\{\delta \in S : \lambda \cap \bigcup_{x \in \mathcal{P}_\delta} N_x^* \in M^*\} = \theta \bmod \text{id}^a(\bar{C})$. For $i \in S$

clearly $x \subseteq y$ (or $x <_{\mathcal{P}_i} y$) $\Rightarrow N_x^* \prec N_y^*$ and \mathcal{P}_i is directed (by the partial order \subseteq or $<_{\mathcal{P}_i}$ recalling clause (vii) of \otimes of Definition 2.4) hence $N'_i := \cup\{N_x^* : x \in \mathcal{P}_i\}$ is $\prec (\mathcal{H}(\chi), \in, <_\chi^*)$ and even $\prec N_i$ and N'_i has cardinality $< \kappa$ (as $|\mathcal{P}_i| < \kappa$ and each N_x^* has cardinality $< \kappa$ and κ is regular) and we have to show that $\{i \in S : \lambda \cap N'_i \in M^*\} = \theta \bmod \text{id}^a(\bar{C})$.

For each $i \in S$ by the choice of \mathcal{Q} , there is a set a_i such that $N'_i \cap \lambda = (\bigcup_{y \in \mathcal{P}_i} N_y^*) \cap \lambda \subseteq a_i \in \mathcal{P}$; so as \mathcal{P} and $\langle N_y^* : y \in \mathcal{P}_i \rangle$ belong to N_{i+1} , see clause (ix) of Definition 2.4 without loss of generality $a_i \in N_{i+1}$. Let $\mathfrak{a}_i =: \text{Reg} \cap a_i \cap \lambda^+ \setminus \theta^+$, so \mathfrak{a}_i is a set of $< \kappa$ regular cardinals $\geq \theta^+$ and $\mathfrak{a}_i \in N_{i+1}$ too, so there is a generating sequence $\langle \mathfrak{b}_\lambda[\mathfrak{a}_i] : \lambda \in \text{pcf}(\mathfrak{a}_i) \rangle$ as in [Sh:g, VII, 2.6] = [Sh 371, 2.6], without loss of generality it is definable from \mathfrak{a}_i (in $(\mathcal{H}(\chi), \in, <_\chi^*)$ say the $<_\chi^*$ -first such object). Also $\mathfrak{a}_i \in \mathcal{P} \subseteq M^*$ so $\mathfrak{a}_i \in M^*$. As $\mathfrak{a}_i \in N_{i+1}$ we have $\langle \mathfrak{b}_\lambda[\mathfrak{a}_i] : \lambda \in \text{pcf}(\mathfrak{a}_i) \rangle \in N_{i+1} \cap M^*$, and also there is $\langle f_{\partial, \alpha}^{\mathfrak{a}_i} : \alpha < \partial, \partial \in \text{pcf}(\mathfrak{a}_i) \rangle$ as in [Sh:g, VIII, 1.2] = [Sh 371, 1.2], and again without loss of generality it belongs to $N_{i+1} \cap M^*$. As $\max \text{pcf}(\mathfrak{a}_i) \leq \text{cov}(\lambda, \kappa, \kappa, 2) = \mu(1)$, (first inequality by [Sh:g, II, 5.4] = [Sh 355, 5.4]) clearly each $f_{\partial, \alpha}^{\mathfrak{a}_i} \in M^*$.

Let

$$\odot_1 \quad h \text{ be the function with domain } \mathfrak{a} := \bigcup_{i \in S} \mathfrak{a}_i \text{ defined by } h(\sigma) = \sup(\sigma \cap \bigcup_{i < \theta} N_i).$$

So by [Sh:g, VIII, 2.3](1) = [Sh 371, 2.3](1)

- ⊙₂ if $i \in S$ then $h \restriction \mathfrak{a}_i$ has the form $\text{Max}\{f_{\partial_\ell, \alpha_\ell}^{\mathfrak{a}_i} : \ell < n\}$ for some $n < \omega$, $\partial_\ell \in \text{pcf}(\mathfrak{a}_\ell)$ and $\alpha_\ell < \partial_\ell$ for $\ell < n$
hence
- ⊙₃ if $i \in S$ then $h \restriction \mathfrak{a}_i$ belongs to M^*
and obviously (as $\sigma \in \mathfrak{a}_i \wedge i < j_1 < j_2 \Rightarrow \sup(\sigma \cap N_{j_1}) < \sup(\sigma \cap N_{j_2})$)
- ⊙₄ $\sigma \in \text{Dom}(h) \Rightarrow \text{cf}(h(\sigma)) = \theta$.

Let e be a definable function in $(\mathcal{H}(\chi), \in, <_\chi^*, \lambda, \kappa)$ with $\text{Dom}(e) = \lambda + 1$ such that $e(\alpha) = e_\alpha$ is a club of α of order type $\text{cf}(\alpha)$, enumerated as $\langle e_\alpha(\zeta) : \zeta < \text{cf}(\alpha) \rangle$.

Now for each $\sigma \in \bigcup_{i < \theta} \mathfrak{a}_i$ let

- ⊙₅ $E_\sigma =: \{i < \theta : (\forall \zeta < \theta)[e_{h(\sigma)}(\zeta) \in N_i \Leftrightarrow \zeta < i], i \text{ is a limit ordinal and } \sup(N_i \cap \sigma) = \sup\{e_{h(\sigma)}(\zeta) : \zeta < i\}\}$.

Clearly E_σ is a club of θ , hence (on $\langle b_j^* : j < \theta \rangle$, see clause (viii)⁻ of Definition 2.1)

$$E = \{\delta < \theta : \delta \text{ is a limit ordinal and } \sigma \in \bigcup \{\mathfrak{a}_i : i < \delta\} \subseteq \text{Reg} \cap \lambda^+ \setminus \theta^+ \Rightarrow \delta \in \text{acc}(E_\sigma) \text{ and } N_\delta \cap \theta = \delta\}$$

is a club of θ . For each $\delta \in E \cap S$ such that $C_\delta \subseteq E$, let $\delta^* := \sup(\kappa \cap N'_\delta) = \sup(\kappa \cap \bigcup_{y \in \mathcal{P}_\delta} N_y^*)$ so $\delta^* < \kappa$, and we define by induction on n models $M_{y, \delta, n}$ for every $y \in \mathcal{P}_\delta$, (really, they do not depend on δ).

First, $M_{y, \delta, 0}$ is the Skolem Hull in M_λ^* of $\{i : i \in y\} \cup (N'_\delta \cap \kappa)$.
Second, $M_{y, \delta, n+1}$ is the Skolem Hull in M_λ^* of $M_{y, \delta, n} \cup \{e_{h(\sigma)}(\zeta) : \sigma \in (\text{Reg} \cap \lambda^+ \setminus \theta^+) \cap M_{y, \delta, n} \text{ and } \zeta \in y\}$. Now we note

- (*)₀ if $y \in \{b_i^* : i < \zeta\}$, $\zeta \in C_\delta$ and $\delta \in E$ then $N_y^* \in N_\zeta$ hence $N_y^* \prec N_\zeta$.

[Why? By clause (ix) of \otimes of Definition 2.4 we have $N_y^* \in N_\zeta$; as $\|N_y^*\| < \kappa < \theta$ and $N_\zeta \cap \theta \in \theta$ as $\zeta \in C_\zeta \subseteq E$ we have $N_y^* \subseteq N_\zeta$ hence $N_y^* \prec N_\zeta$.]

- (*)₁ if $\zeta \in E (\subseteq \theta)$ and $\sigma \in \text{Reg} \cap N_\zeta \cap \lambda^+ \setminus \theta$ then $e_{h(\sigma)}(\zeta) = \sup(N_\zeta \cap \sigma)$.

[Why? By the choice of E .]

- (*)₂ assume $\delta \in S$ satisfies $\delta \in E$, moreover $C_\delta \subseteq E$; if $y \in \mathcal{P}_\delta$ and $\sigma \in N_y^* \cap \text{Reg} \cap \lambda^+ \setminus \theta^+$ then $(h(\sigma))$ has cofinality θ , the sequence $\langle e_{h(\sigma)}(\zeta) : \zeta < \theta \rangle$ is increasing continuous with limit $h(\sigma)$ and):

- (i) if $y \in \{b_i^* : i < \zeta\}$ and $\zeta \in C_\delta$ then $\sup(N_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$

- (ii) if $y \in \{b_i^* : i < \zeta\}, \zeta \in z \in \mathcal{P}_\delta$ and $y <_{\mathcal{P}_\delta} z$ then $y \in N_z, N_y^* \in N_z^*, N_y^* \prec N_z^*$ and $e_{h(\sigma)}(\zeta) \in N_z^*$
- (iii) $\{e_{h(\sigma)}(\zeta) : \zeta \in C_\delta\}$ is a subset of $N'_\delta = \bigcup_{z \in \mathcal{P}_\delta} N_z^*$
- (iv) the set above is an unbounded subset of $N'_\delta \cap \sigma$.

[Why? Clause (i): So we assume $\zeta \in C_\delta$ and $y \in \{b_i^* : i < \zeta\}$.

By $(*)_0$ we have $N_y^* \prec N_\zeta$. By the definition of E_σ as $\sigma \in N_y^* \prec N_\zeta \wedge \zeta \in E$ clearly $\zeta \in E_\sigma$ hence $\sup(N_\zeta \cap \theta) = e_{h(\sigma)}(\zeta)$ by $(*)_1$.

Clause (ii): So assume $y \in \{b_i^* : i < \zeta\}, \zeta \in z$ and $y <_{\mathcal{P}_\delta} z$ (so $y, z \in \mathcal{P}_\delta$) hence $\mathcal{P}_{z,\zeta} = \{x \in \bigcup_{\alpha \in S} \mathcal{P}_\alpha : x \subseteq z \cap \zeta\}$ has cardinality $< \kappa$ and $z \cap \zeta \in N_z^*$ by clause

(x) of 2.4, so $\mathcal{P}_{z,\zeta} = \{x \in \bigcup \{\mathcal{P}_\alpha : \alpha < \delta\} : x \subseteq z \cap \zeta\} \in N_z^*$, so (as $N_z^* \cap \kappa \in \kappa$, $|\mathcal{P}_{z,\zeta}| < \kappa$) clearly $\mathcal{P}_{z,\zeta} \subseteq N_z^*$ hence $y \in N_z^*$. By clause (viii) of \otimes of Definition 2.4 it follows that $N_y^* \in N_z^*$. But $|N_y^*| < \kappa \wedge N_z^* \cap \kappa \in \kappa$ hence $N_y^* \subseteq N_z^*$ so $N_y^* \prec N_z^*$. But $\sigma \in N_y^*$ hence $\sigma \in N_z^*$. Also $N_\zeta \in N_z^*$ as $\zeta \in z \subseteq N_z^*$ recalling (viii) of 2.4 hence $e_{h(\sigma)}(\zeta) = \sup(N_\zeta \cap \sigma) \in N_z^*$ recalling $(*)_1$ so we have shown all clauses of (ii).

Clause (iii): So let $\zeta \in C_\delta$; by clause (vii)(β) of Definition 2.1 we know that $C_\delta = \bigcup \{y : y \in \mathcal{P}_\delta\}$ hence for some $y_1 \in \mathcal{P}_\delta$ we have $\zeta \in y_1$. By clause (x) of \otimes from Definition 2.4 we have $y_1 \subseteq N_{y_1}^*$ hence $\zeta \in N_{y_1}^*$. Also we are assuming in $(*)_2$ that $\sigma \in N_y^*, y \in \mathcal{P}_\delta$, so recalling \mathcal{P}_δ is directed, we can find $y_2 \in \mathcal{P}_\delta$ which is a common \subseteq -upper bound of y_1, y_2 hence $N_y^* \prec N_{y_2}^*, N_{y_1}^* \prec N_{y_2}^*$ hence $\sigma, \zeta \in N_{y_2}^*$.

By the choice of the function e and the model M_λ^* clearly $e(-, -)$ is a function of M_λ^* , but the object \mathbf{x} belongs to $N_{y_2}^*$ and by its choice this implies that $e \in N_{y_2}^*$. By clause (viii) of 2.4 recalling $\zeta \in N_{y_2}^*$ we know that $N_\zeta \in N_{y_2}^*$ but $\sigma \in N_{y_2}^*$ hence by $(*)_1$ we have $\sup(N_\zeta \cap \sigma) \in N_{y_1}^*$. But we are assuming in $(*)_2$ that $C_\delta \subseteq E$ and, see above, $\zeta \in C_\delta$ so $\zeta \in E$ and $\zeta \in C_\delta \subseteq N_\zeta, \sigma \in N_{y_2}^* \subseteq N'_\delta \subseteq N_\zeta$ so $\sup(N_\zeta \cap \sigma) = e_{h(\sigma)}(\zeta)$ so by the previous sentence $e_{h(\sigma)}(\zeta) \in N_{y_2}^*$, hence $e_{h(\sigma)}(\zeta) \in \bigcup \{N_x^* : x \in \mathcal{P}_\delta\} = N'_\delta$ as required.

Clause (iv): By clause (iii) it is $\subseteq N'_\delta$, and by the choice of the function e it is $\subseteq \sigma$ hence it is $\subseteq N'_\delta \cap \sigma$. Now $N'_\delta = \bigcup \{N_z^* : z \in \mathcal{P}_\delta\}$ and $z \in \mathcal{P}_\delta \Rightarrow N_z^* \prec N_\delta$ by $(*)_0$ hence $N'_\delta \subseteq N_\delta$. Now we know that $\langle e_{h(\sigma)}(\zeta) : \zeta < \delta \rangle$ is increasing with limit $e_{h(\sigma)}(\delta) = \sup(N_\delta \cap \sigma)$ hence is unbounded in it and even $\langle e_{h(\sigma)}(\zeta) : \zeta \in C_\delta \rangle$ is an unbounded subset of $e_{h(\sigma)}(\delta)$ and it is included in N'_δ as required.

So $(*)_2$ indeed holds.

Now (A), (B), (C), (D), (E) below clearly suffice to finish.

(A) (a) for $\delta \in S, y \in \mathcal{P}_\delta$ and $n < \omega$ we have $M_{y,\delta,n} \subseteq N'_\delta = \bigcup_{z \in \mathcal{P}_\delta} N_z^*$.

[Why? We prove this by induction on n . First assume $n = 0$, $M_{y,\delta,n}$ is the Skolem hull of $y \cup (N'_\delta \cap \kappa)$ in the model M_λ^* , well defined as $y \subseteq \lambda$ hence $y \subseteq M_\lambda^*$ and $N' \cap \kappa \subseteq \kappa \subseteq \lambda$. As $y \subseteq N_y^* \subseteq N'_\delta$ and $M_\lambda^* \in N_y^* \subseteq N'_\delta$ clearly $M_{y,\delta,n} \subseteq N'_\delta$. Second, assume $n = m + 1$ and $M_{y,\delta,m} \subseteq N'_\delta$. Now $M_{y,\delta,n}$ in the Skolem hull of $M_{y,\delta,m} \cup \{e_{h(\sigma)}(\zeta) : \sigma \in M_{y,\delta,m} \cap \text{Reg} \cap (\lambda^+ \setminus \theta^+) \text{ and } \zeta \in y\}$, so it is enough to show that: if $\sigma \in M_{y,\delta,m}$ (hence $\sigma \in N'_\delta$) and $\sigma \in \text{Reg} \cap \lambda^+ \setminus \theta^+$ and $\zeta \in y$ then $e_{h(\sigma)}(\zeta) \in N'_\delta$. But by $(*)_2(iii)$ this holds.

(b) for $z \subseteq y$ in \mathcal{P}_δ we have $M_{z,\delta,n} \subseteq M_{y,\delta,n}$.

[Why? Just by their choice, i.e. we prove this by induction on $n < \omega$.]

(c) for $y \in \mathcal{P}_\delta$ and $m \leq n$ we have $M_{y,\delta,m} \subseteq M_{y,\delta,n}$.

[Why? Just by their choice, i.e. we prove this by induction on n .]

(d) $M'_\delta := \bigcup \{M_{y,\delta,n} : y \in \mathcal{P}_\delta \text{ and } n < \omega\}$ is $\prec N'_\delta$.

[Why? By the above.]

(e) if $\zeta \in z$ (hence $\zeta \in C_\delta \subseteq E$), $\{y, z\} \subseteq \mathcal{P}_\delta$, $\sup(y) < \zeta, y <_{\mathcal{P}_\delta} z$ and $\sigma \in \text{Reg} \cap \lambda^+ \setminus \theta^+$ then: $\sigma \in N_y^* \prec N_\zeta \Rightarrow e_{h(\sigma)}(\zeta) = \sup(\sigma \cap N_\zeta) \in N_z^*$.

[Why? By $(*)_2(i) + (ii)$ this holds.]

(B) We can also prove that $\langle M_{y,\delta,n} : n < \omega, y \in \mathcal{P}_\delta \rangle$ is definable in $(\mathcal{H}(\chi), \in, <_\chi^*)$ from the parameters $\delta, M_\lambda^*, (\bar{C}, \bar{\mathcal{P}})$ and $h \upharpoonright a_i$, all of them belong to M^* , hence the sequence, and $M'_\delta = \bigcup \{M_{y,\delta,n} : n < \omega, y \in \mathcal{P}_\delta\}$, belongs to M^* .

(C) $M'_\delta \cap \text{Reg} \cap (\theta, \lambda^+)$ is a subset of \mathfrak{a}_δ .

[Why? Use (A)(a) and definition of a_i, \mathfrak{a}_i .]

(D) if $\sigma \in M'_\delta$ and $\sigma \in \text{Reg} \cap \lambda^+ \setminus \kappa$ then $\sigma \cap M'_\delta$ is unbounded in $\sigma \cap N'_\delta$.

[Why? When $\sigma > \theta$ use $(*)_2(iii), (iv)$. For $\sigma = \theta$ we have $N'_\delta \cap \theta \subseteq N_\delta \cap \theta = \delta$ as $\delta \in E$ and $C_\delta \subseteq \delta = \sup(C_\delta)$ so it is enough to show $C_\delta \subseteq N'_\delta$, but C_δ is equal to $\bigcup_{y \in \mathcal{P}_\delta} y$. For $\sigma = \kappa$ see the choice of $M_{y,\delta,0}$. So as $\theta = \kappa^+$ we are done.]

(E) $M'_\delta \cap \lambda = N'_y \cap \lambda$.

[Why? By (A)(a) we have one inclusion, the \subseteq . By the choice of M_λ^* and clause (D) the result follows by [Sh 400, 3.3A, 5.1A] recalling $N'_\delta \cap \kappa \in \kappa$.] $\square_{2.6}$

But to get normality of the filter we better define

2.9 Definition. Assume $\theta = \text{cf}(\theta) > \kappa = \text{cf}(\kappa) > \aleph_0$, $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ and X is a set, of cardinality $\geq \theta$ for simplicity and let χ be large enough. We define a filter $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[X]$ on $[X]^{<\kappa}$ as the set of $Y \subseteq [X]^{<\kappa}$ such that for some $\mathbf{x} \in \mathcal{H}(\chi)$, for every sequence $\langle N_\alpha, N_a^* : \alpha < \theta, a \in \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle$ satisfying \otimes below, there is $A \in \text{id}^a(\bar{C})$

such that $\mathbf{x} \in \bigcup_{a \in \mathcal{P}_\delta} N_a^*$ & $\delta \in S(\bar{C}) \setminus A \Rightarrow \bigcup_{a \in \mathcal{P}_\delta} N_a^* \cap X \in Y$ where

\otimes as in Definition 2.4 omitting $\mathbf{x} \in N_\alpha$.

2.10 Claim. Let $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$.

- 1) An χ such that $\mathcal{P}(X) \subseteq \mathcal{H}(\chi)$ can serve in Definition 2.9, and $\mathbf{x} = Y$ can serve.
- 2) If X_1, X_2 are sets of cardinality $\lambda \geq \chi$ and f is a one-to-one function from X_1 onto X_2 , then f maps $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X_1)$ onto $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X_2)$.
- 3) If $X_1 \subseteq X_2$ has cardinality $\geq \theta$ then $Y \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[X_1] \Rightarrow \{u \in [X_2]^{<\kappa} : u \cap X_1 \in Y\} \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[X_2]$ and $Y \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X_2) \Rightarrow \{u \cap X_1 : u \in Y\} \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X_1)$.
- 2) For any set X of cardinality $\geq \kappa$, really $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X)$ is a fine normal filter on X , i.e.:

- (a) *fine*: $t \in X \Rightarrow \{u \in [X]^{<\kappa} : t \in u\} \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X)$
- (b) *normal*: if $Y_t \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(X)$ for $t \in X$ then $Y := \Delta\{Y_t : t \in X\} = \{u \in [X]^{<\kappa} : u \neq \emptyset \text{ and } t \in u \Rightarrow u \in Y_t\}$.

Proof. 1), 2) Easy.

3) The “fine” is trivial and for normal let \mathbf{x}_t be a witness for $Y_t \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[X]$ now $\mathbf{x} = \langle \mathbf{x}_t : t \in X \rangle$ witness that $Y \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[X]$.

2.11 Claim. Let $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$.

- 1) $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda) \supseteq \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[\lambda]$.
- 2) In 2.6 we can replace $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda)$ by $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[\lambda]$.
- 3) Assume that $\text{cf}(\lambda) \geq \kappa$ and $\beta < \alpha \Rightarrow \lambda > \text{cov}(|\beta|, \kappa, \kappa, 2)$. Then there is $S \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda)$ such that $\alpha < S \Rightarrow \lambda > |\{u \in S : u \subseteq \alpha\}|$.

Proof. 1) Trivial.

2) Repeat the proof, the change is minor.

3) We can find $\mathcal{Q} = \{u_i : i < \lambda\} \subseteq [\lambda]^{<\kappa}$ which is cofinal such that $\forall \alpha < \lambda(\beta)(\alpha)[\alpha \leq \beta < \lambda \wedge \{\{u_i : i < \beta, u_i \subseteq \alpha\}\}]$ is cofinal in $[\alpha]^{<\kappa}$.

2.12 Remark. In 2.6 we can replace $\theta = \kappa^+$ by $\theta > \kappa_\sigma > \sigma = \text{cf}(\sigma)$ and $\alpha < \theta \Rightarrow |\alpha|^{<\sigma > \text{tr}} < \theta$ and $\delta \in S(\bar{C}) \Rightarrow \text{cf}(\delta) = \sigma$.

Proof. Fill.

2.13 Conclusion. Suppose $\lambda > \kappa > \aleph_0$ are regular cardinals and $(\forall \mu < \lambda)[\text{cov}(\mu, \kappa, \kappa, 2) < \lambda]$.

1) If for $\alpha < \lambda$, a_α is a subset of λ of cardinality $< \kappa$ and $S \in \mathcal{D}_{<\kappa}(\lambda)$ and $T_1 \subseteq \{\delta < \lambda : \text{cf}(\delta) \geq \kappa\}$ is stationary, then we can find a stationary $T_2 \subseteq T_1, c \subseteq \lambda$ and $\langle b_\delta : \delta \in T \rangle$ such that:

$$a_\delta \subseteq b_\delta \in S \text{ for } \delta \in T_2$$

$$b_\delta \cap \delta = c \text{ for } \delta \in T_2.$$

2) If in addition $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\kappa^+, \kappa]$ and $S \in (\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}(\lambda))^+$ then part (1) holds for this S .

Remark. See on this and on 2.15 Rubin Shelah [RuSh 117, 4.12, pg.76] and [Sh 371, §6]. There we do not know that $(\forall \mu < \lambda)[\text{cov}(\mu, \kappa, \kappa, 2) < \lambda]$ implies (as proved ehre) that

$\boxtimes_{\lambda, \kappa}$ for each $\alpha < \lambda$ we can find S_α a stationary $S_\alpha \subseteq [\alpha]^{<\lambda}$ of cardinality $< \lambda$; moreover such that $\{\alpha\} \cup u : u \in S_\alpha, \alpha < \lambda\} \subseteq [\lambda]^{<\kappa}$ is stationary, (if λ is a successor cardinal, the moreover follows. So the assumption there seems just what was used now. So we could just quote.

Proof. 1) By part (2).

2) For each $\alpha < \lambda$ let $S_\alpha \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[\alpha]$ be of cardinality $\text{cov}(|\alpha|, \kappa, \kappa, 2)$.

Let $S = \{u \in [\lambda]^{<\kappa} : \text{if } \alpha \in u \setminus \kappa^+ \text{ then } u \cap \alpha \in S_\alpha\}$, so by 2.10 we know that $S \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}[\lambda]$; and by 2.11(3) without loss of generality

$$(*) \quad \alpha < \lambda \Rightarrow \{u \in S : u \subseteq \alpha\} \text{ has cardinality } < \lambda.$$

Now for each $\alpha < \lambda$ let $b_\alpha \in S$ be such that $a_\alpha \subseteq b_\delta$, clearly exist and let $h : T_1 \rightarrow \lambda$ be defined by $h(\delta) = \sup(b_\delta \cap \delta)$ so $\delta \in T_2 \Rightarrow h(\delta) < \delta$ as $\text{cf}(\delta) \geq \kappa > |b_\delta|$. So for some $\gamma_* < \gamma$ the set $T'_2 := \{\delta \in T_1 : h(\delta) = \gamma_*\}$ is stationary and by (*) for some c the set $T_2 := \{\delta \in T'_2 : b_\delta \cap \delta = c\}$ is stationary. Let $E = \{\delta < \lambda : \delta \text{ is a limit ordinal such that } \alpha < \delta \Rightarrow b_\alpha \subseteq \delta\}$, it is a club of λ .

2.14 Conclusion. If $\lambda > \kappa > \aleph_0$, λ and κ are regular cardinals and $[\kappa < \mu < \lambda \Rightarrow \text{cov}(\mu, \kappa, \kappa, 2) < \lambda]$ then $\{\delta < \lambda : \text{cf}(\delta) < \kappa\} \in \check{I}[\lambda]$.

Proof. Use $\mu(3)$ of 2.6.

2.15 Claim. Let $(*)_{\mu, \lambda, \kappa}$ mean: if $a_i \in [\lambda]^{<\kappa}$ for $i \in S$ and $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \kappa\}$ is stationary, then for some $b \in [\lambda]^{<\kappa}$ the set $\{i \in S : a_i \cap i \subseteq b\}$ is stationary. Let $(*)_{\mu, \lambda, \kappa}^-$ be defined similarly but $\{i \in S : a_i \subseteq b\}$ only unbounded. Then for $\aleph_0 < \kappa < \lambda < \mu$ regular we have:

$$\begin{aligned} \text{cov}(\lambda, \kappa, \kappa, 2) < \mu &\Rightarrow (*)_{\mu, \lambda, \kappa} \Rightarrow (*)_{\mu, \lambda, \kappa}^- \\ &\Rightarrow (\forall \lambda') [\kappa < \lambda' \leq \lambda \ \& \ \text{cf}(\lambda') < \kappa \Rightarrow \text{pp}_{<\kappa}(\lambda') < \mu]. \end{aligned}$$

Remark. So it is conceivable that the \Rightarrow are \Leftrightarrow . See [Sh 430, §3].

Proof. Straightforward. □_{2.15}

Exercise: Generalize to the following filter.

Let $\theta = \text{cf}(\theta) \geq \kappa = \text{cf}(\kappa)$ and $S_* \subseteq [\theta]^{<\kappa}$ be stationary. For any set X of cardinality $\geq \theta$ we define a filter $\mathcal{D}_{S_*}^1[X]$ as follows: $Y \in \mathcal{D}_{S_*}^1[X]$ iff $Y \subseteq [X]^{<\kappa}$ and for any χ large enough there is $\mathbf{x} \in \mathcal{H}(\chi)$ such that if $\langle N_\alpha, f_\alpha : \alpha \leq \theta \rangle$ satisfy \circledast below, then for some $S' \in \mathcal{D}_{<\kappa}(\theta)$ for every $u \in S_* \cap S'$ we have:

if $\mathbf{x} \in f''_\theta(u)$ then $f''(u) \in Y$, when:

- \circledast (a) $N_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$
- (b) N_α is \prec -increasing continuous
- (c) $\|N_\alpha\| < |\alpha|^+ + \theta$
- (d) $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$ if $\alpha < \theta$
- (e) can add $\langle \kappa, \theta, X, S_* \rangle \in N_0$.

§3 NICE FILTERS REVISITED

This generalizes [Sh 386] (and see there).
See [Sh 410, §5] on this generalization of normal filters.

3.1 Convention. 1) \mathbf{n} is a niceness context; we use κ , FILL , etc., for $\kappa_{\mathbf{n}}$, $\text{Fil}_{\mathbf{n}} = \text{FIL}(\mathbf{n})$ when dealing from the content.

3.2 Definition. We say the \mathbf{n} is a niceness context or a κ -niceness context or a (κ, μ) -niceness context if it consists of the following objects satisfying the following conditions:

- (a) κ is a regular uncountable cardinal
- (b) $I \subseteq {}^{\omega}>\omega$ is non-empty \triangleleft -downward closed with no \triangleleft -maximal member²
default value is $\{0_n : n < \omega\}$
- (c) let μ be $> \kappa$ and $\langle \mathcal{Y} : i < \kappa \rangle$ is a sequence of pairwise disjoint sets and $\mathcal{Y} \cup \{\mathcal{Y}_i : i < \omega_1\}$ so $i < \omega_1 \Rightarrow |\mathcal{Y}|, |\mathcal{Y}_i|$
- (d) the function ι with domain \mathcal{Y} is defined by $\iota(y) = i$ when $y \in \mathcal{Y}_i$
- (e) \mathbf{e} is a set of equivalence relations e on \mathcal{Y} refining $\bigcup_{i < \omega_1} \mathcal{Y}_i \times \mathcal{Y}_i$ with $< \mu^*$
equivalence classes, each class of cardinality $|\mathcal{Y}|$
- (f) for $e \in \mathbf{e}$, $\text{FIL}(e) = \text{FIL}(e, \mathbf{n})$ is a set of D such that:
 - (α) D is a filter on \mathcal{Y}/e ,
 - (β) for any club C of κ we have $\bigcup_{i \in C} \mathcal{Y}_i/e \in D$,
 - (γ) normality: if $X_i \in D$ for $i < \omega_1$ then the following set belongs to D :
 $\{(\delta, j)/e : (\delta, j) \in \mathcal{Y}, \delta \text{ limit and } i < \delta \Rightarrow (\delta, j) \in X_i\}$
- (g) $\text{Suc} \in \{(D_1, D_2) : e(D_1) \leq e(D_2)\}$.

Remark. For \mathbf{e} an important case is when it is a singleton $\{\cup\{\mathcal{Y}_i \times \mathcal{Y}_i : i < \kappa\}\}$, so we are dealing with normal filters on the old case.

²For \mathcal{T} the two interesting cases are $\mathcal{T} = {}^{\omega}>\omega$ and $\mathcal{T} = \{<>\}$ and ${}^{\omega}>\{0\}$. The default value will be ${}^{\omega}>\omega$.

3.3 Definition. Let \mathbf{n} be a κ -niceness context.

- 1) We say $e_1 \leq e_2$ if e_2 refines e_1 . If not said otherwise, every e is from \mathbf{e} . Let \mathbf{e}_μ be the set of all such equivalence relations with $< \mu$ equivalence classes. Let $\iota(x/e) = \iota(x)$.
- 2) $\text{FIL} = \text{FIL}(\mathbf{n})$ is $\bigcup_{e \in \mathbf{e}} \text{FIL}(e, \mathbf{n})$. For $D \in \text{FIL}$, let $e = e[D]$ be the unique $e \in \mathbf{e}$ such that $D \in \text{FIL}(e, \mathbf{n})$.
- 3) For $D \in \text{FIL}(e)$ let $D^{[*]} = \{X \subseteq \mathcal{Y} : X^{[*]} \in D\}$; see (5) below.
- 4) For $D \in \text{FIL}(\mathbf{n})$ and $e(1) \geq e(D)$, let $D^{[e(1)]} = \{X \subseteq \mathcal{Y}/e(1) : X^{[*]} \in D^{[*]}\}$, see (5) below.
- 5) For $A \subseteq \mathcal{Y}/e$, $A^{[*]} = \{(x/e) : (x/e) \in A\}$, and for $e(1) \geq e$ let $A^{[e(1)]} = \{y/e(1) : y/e \in A\}$.

3.4 Definition. 1) For $D \in \text{FIL}(e, \mathbf{n})$, let D^+ be $\{Y \subseteq \mathcal{Y}/e : Y \neq \emptyset \text{ mod } D\}$.

- 2) \mathbf{n} is 1-closed if $D \in \text{FIL}(\mathbf{n})$, $A \in D^+ \Rightarrow D + A \in \text{FIL}(\mathbf{n})$.
- 3) \mathbf{n} is 0-closed if for every $D_1 \in \text{FIL}_{\mathbf{n}}$ and $A \in D_1^+$ there is $D_2 \in \text{FIL}_2$ such that $(D_1 + A) \in (D_2) \subseteq D_2$.
- 4) A niceness context \mathbf{n} is full if
 - (a) for every $e \in \mathbf{e}_{\mathbf{n}}$, every filter on $\mathcal{Y}_{\mathbf{n}}/e$ which is normal (with respect to the function $\iota_{\mathbf{n}}$) belong to $\text{FIL}_{\mathbf{n}}(e)$.

4A) A niceness content \mathbf{n} is semi-full when: for every $e_1 \in \mathbf{e}_{\mathbf{n}}$ and $D_1 \in \text{FIL}_{\mathbf{n}}(e_1)$ and $e_2, e_1 \leq e_2 \in \mathbf{e}_{\mathbf{n}}$ and $\mathcal{A} \subset \mathcal{P}(\mathcal{Y}_{\mathbf{n}}/e_2)$ $\text{lift}(W) \in \text{FIL}(e_2)$ whenever

- (*) _{e_1, e_2, D_1, W}
 - (a) $e_1 \leq e_2$ in $\mathbf{e}_{\mathbf{n}}$
 - (b) $D_1 \in \text{FIL}_{\mathbf{n}}(e_2)$
 - (c) $\mu \geq 2^{(\mathcal{Y}/e_2)}$ (or more ???)
 - (d) $W \subseteq [\mu]^{\leq \aleph_0}$ is stationary
 - (e) $D_2 = \text{lift}(W, D_1^{[e_2]})$ is normal (i.e. $\emptyset \in \text{lift}(W, D_1)$).

5) A niceness context \mathbf{n} is thin when

$$\text{Suc}_{\mathbf{n}} = \{(D_1, D_2) : D_1 = D_2 \in \text{FIL}_{\mathbf{n}} \text{ and } D_2 = D_1^{[e_1]} + A \text{ for some } A \in (D_1^{[e_1]})^+\}.$$

6) A niceness context \mathbf{n} is thick if: $\text{Suc}_{\mathbf{n}} = \{(D_1, D_2) : D_1, D_2 \in \text{FIL}_{\mathbf{n}}, e(D_1) \leq e(D_2) \text{ and } D_1^{[e_2]} \subseteq D_2 \text{ and if } \mu = 2^{|\mathcal{Y}_{\mathbf{n}}/e_2|}, W_1 \subseteq [\mu]^{\leq \aleph_0} \text{ is stationary and } \text{lift}(W, D_1) = D_1 \text{ then for some stationary } W_2 \subseteq W_1 \text{ we have } \text{lift}(W_2, D_2) = D_2\}$.

Remark. 1) On lift see Definition 3.17, HERE??
 2) We can use more freedom in the higher objects.

3.5 Claim. *Assume*

- (a) *the κ -niceness context is thick*
- (b) $D_1 \in \text{FIL}_{\mathbf{n}}(e_1)$
- (c) $e_1 \leq e_2 \in \mathbf{e_d}$
- (d) *for each $y \in \mathcal{Y}_{\mathbf{n}}/e_1$, $\langle z_{y,\varepsilon} : \varepsilon < \varepsilon_y \rangle$ list $\{z/e_2 : z \in y_1\}$, $d_{y,\varepsilon}$ is a κ -complete filter on ε_y*
- (e) $D_2 \in \text{FIL}_{\mathbf{n}}(e_2)$
- (f) *if $A \in D_2$ then $\{y \in \mathcal{Y}_{\mathbf{n}}/e_1 : \{\varepsilon < \varepsilon_y : z_{y,\varepsilon} \in A\} \in d_{y,\varepsilon}\}$ belongs to D_1 .*

Then $D_2 \in \text{Suc}_{\mathbf{n}}(D_1)$.

Discussion: We may consider allowing player I , in the beginning of each move to choose W_n as above.

3.6 Definition. (0) For $f : \mathcal{Y}/e \rightarrow X$ let $f^{[*]} : \mathcal{Y} \rightarrow X$ be $f^{[*]}(x) = f(x/e)$. We say $f : \mathcal{Y} \rightarrow X$ is supported by e if it has the form $g^{[*]}$ for some $g : \mathcal{Y}/e \rightarrow X$. If $e_1, e_2 \in \mathbf{e}$ and $f_\ell : \mathcal{Y}/e_\ell \rightarrow X$ for $\ell = 1, 2$ then: we say $f_1 = f_2^{[e_1]}$ if $f_1^{[*]} = f_2^{[*]}$. Writing $f^{[*]}$ for $f \in {}^{\omega_1}X$ we identify $\{i\}$, $i < \omega_1$ with \mathcal{Y}_i .

(1) Let $F_c(\mathcal{T}, e) = F_c(\mathcal{T}, e, \mathcal{Y})$ be the family of \bar{g} , a sequence of the form $\langle g_\eta : \eta \in u \rangle$, $u \in f_c(\mathcal{T}) =$ the family of non-empty finite subsets of ${}^{\omega}>\omega$ closed under taking initial segments, and for each $\eta \in u$ we have $g_\eta \in {}^{\mathcal{Y}}\text{Ord}$ is supported by e . Let $\text{Dom}(\bar{g}) = u$, $\text{Range}(\bar{g}) = \{g_\eta : \eta \in u\}$. We let $e = e(\bar{g})$, for the minimal possible e assuming it exists and we shall say $g_\eta <_D g_\nu$ instead $g_\eta <_{D^{[*]}} g_\nu$ and not always distinguish between $g \in {}^{\mathcal{Y}/e}\text{Ord}$ and $g^{[*]}$ in an abuse of notation.

(2) We say \bar{g} is decreasing for D or D -decreasing (for $D \in \text{FIL}(e, I)$) if $\eta \triangleleft \nu \Rightarrow g_\nu <_D g_\eta$.

(3) If $u = \{<>\}$, $g = g_{<>}$ we may write g instead $\langle g_\eta : \eta \in u \rangle$.

3.7 Definition. 1) For $e \in \mathbf{e}$, $D \in \text{FIL}(e)$ and D -decreasing $\bar{g} \in F_c(\mathcal{T}, e)$ we define a game $\mathfrak{D}^*(D, \bar{g}, e) = \mathfrak{D}^*(D, \bar{g}, e, \mathbf{n})$. In the n th move (stipulating $e_{-1} = e$, $D_{-1} = D$, $\bar{g}_{-1} = \bar{g}$):

the case \mathbf{n} is then

player I chooses $e_n \geq e_{n-1}$ and $A_n \subseteq \mathcal{Y}/e_n$, $A_n \neq \emptyset \bmod D_{n-1}^{[e_n]}$ and he chooses $\bar{g}^n \in F_c(\mathcal{T}, e_n)$ extending \bar{g}_{n-1} (i.e. $\bar{g}^{n-1} = \bar{g}^n \upharpoonright \text{Dom}(\bar{g}_{n-1})$), \bar{g}^n supported by e_n and \bar{g}^n is $(D_n^{[e_n]} + A_n)$ -decreasing, player II chooses $D_n \in \text{FIL}(e_n)$ extending $D_{n-1}^{[e_n]} + A_n$.

In the general case:

Player I chooses e_n and $D_{n,1} \in \text{Duc}_{\mathbf{n}}(D_{n-1})$ and let $e_n = e(D_{n-1})$ and he chooses $\bar{g}^n \in F \subset (\mathcal{T}, e(D_{n-1}))$ which is extending \bar{g}^{n-1} then $\eta \in \text{Dom}(\bar{g}^n)$ (i.e. $\bar{g}^{n-1} = \bar{g}^n \upharpoonright \text{Dom}(\bar{g}^{n-1})$), \bar{g}^n supported by $e(D_{n,1})$ and \bar{g}^n is $D_{n,1}$ -decreasing.

Player II chooses $D_n = D_{n,2} \in \text{FIL}(\mathbf{e}_n)$ extending $D_{n,1}$.

In the end, the second player wins if $\bigcup_{n < \omega} \text{Dom}(\bar{g}^n)$ has no infinite branch.

2) Let $\bar{\gamma}$ be such that $\text{Dom}(\bar{\gamma}) = \bigcup_{n < \omega} \text{Dom}(\bar{g})$ and each γ_η is an ordinal decreasing with η . Now $\mathfrak{D}^{\bar{\gamma}}(D, \bar{g}, e)$ is defined similarly to $\mathfrak{D}^*(D, \bar{g}, e)$ but the second player has in addition, to choose an ordinal α_η for $\eta \in \text{Dom}(\bar{g}^n) \setminus \bigcup_{\ell < n} \text{Dom}(\bar{g}^\ell)$ such that

$[\eta \triangleleft \nu \ \& \ \nu \in \text{Dom}(\bar{g}^{n-1}) \Rightarrow \alpha_\nu < \alpha_\eta]$ we let $\alpha_\eta = \gamma_\eta$ for $\eta \in \text{Dom}(\bar{g})$.

3) $w\mathfrak{D}^*(D, \bar{g}, e)$ and $w\mathfrak{D}^{\bar{\gamma}}(D, \bar{g}, e)$ are defined similarly but e is not changed during a play. (If e.g. $\mathbf{e} = \{e\}$ then this makes not difference.)

4) If $\bar{\gamma} = \langle \gamma_{<} \rangle$, $\bar{g} = \langle g_{<} \rangle$ we write $\gamma_{<}$ instead $\bar{\gamma}$, $g_{<}$ instead \bar{g} .

5) If $E \subseteq \text{FIL}$ the games \mathfrak{D}_E^* , $\mathfrak{D}_E^{\bar{\gamma}}$ are defined similarly, but player II can choose filters only from E (so we naturally assume to have $A \in D^+$, $D \in E \Rightarrow D + A \in E$).

3.8 Remark. Denote the above games \mathfrak{D}_0^* , $\mathfrak{D}_0^{\bar{\gamma}}$, $w\mathfrak{D}_0^*$. Another variant is

3) For $e \in \mathbf{e}$, $D \in \text{FIL}(e)$ and D -decreasing $\bar{g} \in F_c(\mathcal{T})$ we define a game $\mathfrak{D}_1^*(D, \bar{g}, e)$. We stipulate $e_{-1} = e$, $D_{-1} = D$.

In the n th move first player chooses $e_n, e_{n-1} \leq e_n \in \mathcal{T}$ and $D'_n \in \text{FIL}(e_n)$ and D'_n -decreasing \bar{g}^n extending \bar{g}^{n-1} such that $(D_{n-1} + A_n)^{[e_n]} \subseteq D_n$ and:

(*) for some $A_n \subseteq \mathcal{Y}/e_{n-1}$, $A_n \neq \emptyset \bmod D_{n-1}$ we have:

(i) D'_n is the normal filter on \mathcal{Y}/e_n generated by $(D_{n-1} + A_n)^{[e_n]} \cup \{A_\zeta^n : \zeta < \zeta_n^*\}$ where for some $\langle C_\zeta : \zeta < \zeta_n \rangle$ we have:

(a) each C_ζ is a club of ω_1 ,

(b) if $\zeta_\ell < \zeta_n^*$ for $\ell < \omega$, $i \in \bigcap_{\ell < \omega} C_{\zeta_\ell}$, $x \in \mathcal{Y}/e_{n-1}$, and $\iota(x) = i$, then for

some $x' \in \mathcal{Y}/e_n$, we have $x' \subseteq x$, $x' \in \bigcap_{\ell < \omega} A_{\zeta_\ell}^n$.

The first player also chooses \bar{g}^n extending \bar{g}^{n-1} , D'_n -decreasing. Then second player chooses D_n such that $D'_n \subseteq D_n \in \text{FIL}(e_n)$.

2) We define $\mathfrak{D}_1^{\bar{\gamma}}(D, \bar{g}, e)$ as in (2) using \mathfrak{D}_1^* instead of \mathfrak{D}_0^* .

3) If player II wins, e.g. $\mathfrak{D}_E^{\bar{\gamma}}(D, \bar{f}, e)$ this is true for $E' =: \{D' \in G : \text{player II wins } \mathfrak{D}_{E'}^{\bar{\gamma}}(D', \bar{f}, e)\}$.

- 3.9 Definition.** 1) We say $D \in \text{FIL}$ is nice to $\bar{g} \in F_c(\mathcal{T}, e, \mathcal{Y})$, $e = e(D)$, if player II wins the game $\mathfrak{D}^*(D, \bar{g}, e)$ (so in particular \bar{g} is D -decreasing, \bar{g} supported by e).
 2) We say $D \in \text{FIL}$ is nice if it is nice to \bar{g} for every $\bar{g} \in F_c(\mathcal{T}, e)$.
 3) We say D is nice to α if it is nice to the constant function α . We say D is nice to $g \in {}^\kappa\text{Ord}$ if it is nice to $g^{[e(D)]}$.
 4) “Weakly nice” is defined similarly but e is not changed.
 5) Above replacing D by \mathbf{n} means: for every $D \in \text{FIL}_{\mathbf{n}}$.

3.10 Remark. “Nice” in [Sh 386] is the weakly nice here, but

- (a) we can use \mathbf{n} with $\mathbf{e}_{\mathbf{n}} = \{e\}$
- (b) formally they act on different objects; but if $xey \Leftrightarrow \iota(x) = \iota(y)$ we get a situation isomorphic to the old one.

3.11 Claim. Let $D \in \text{FIL}$ and $e = e(D)$.

- 1) If D is nice to f , $f \in F_c(\mathcal{T}, e)$, $g \in F_c(\mathcal{T}, e)$ and $g \leq f$ then D is nice to f .
- 2) If D is nice to f , $e = e(D) \leq e(1) \in \mathbf{e}$ then $D^{[e(1)]}$ is nice to $f^{[e(1)]}$.
- 3) The games from 3.7(2) are determined and winning strategies do not need memory.
- 4) D is nice to \bar{g} iff D is nice to $g_{<}$ (when $\bar{g} \in F_c(\mathcal{T}, e)$ is D -decreasing).
- 5) If $\mathbf{e} \subseteq \mathbf{e}$ and for simplicity $\bigcup_{i < \omega_1} \{i\} \times \mathcal{Y}_i \in \mathbf{e}$ and for every $e \in \mathbf{e}$, $e \leq e(1) \in \mathbf{e}$ for some permutation π of $\bar{\mathcal{Y}}$ (i.e. a permutation of \mathcal{Y} mapping each \mathcal{Y}_i ($i < \omega_1$) onto itself) (and \mathbf{n} is full for simplicity) we have $\pi(e) = e$, $\pi(e(1)) \leq e(2) \in \mathbf{e}$ then we can replace \mathbf{e} by \mathbf{e} .
- 6) For $\mathbf{e} = \mathbf{e}_{\mu}$ (where $\mu \leq \mu^*$) there is \mathbf{e} as above with: $|\mathbf{e}|$ countable if μ is a successor cardinal ($> \aleph_1$), $|\mathbf{e}| = \text{cf}(\mu)$ if μ is a limit cardinal.

Proof. Left to the reader. (For part (4) use 3.12(2) below).

3.12 Claim. 1) Second player wins $\mathfrak{D}^*(D, \bar{g}, e)$ iff for some $\bar{\gamma}$ second player wins $\mathfrak{D}^{\bar{\gamma}}(D, \bar{g}, e)$.

- 2) If second player wins $\mathfrak{D}^{\gamma}(D, f, e)$ then for any D -decreasing $\bar{g} \in F_c(\mathcal{T}, e)$, \bar{g} supported by e and $\bigwedge_{\eta, y} g_{\eta}(y) \leq f(y)$, the second player wins in $\mathfrak{D}^{\bar{\gamma}}(D, \bar{g}, e)$, when we let

$$\gamma_{\eta} = \gamma + [\max\{(\ell g(\nu) - \ell g(\eta) + 1) : \nu \text{ satisfies } \eta \trianglelefteq \nu \in \text{Dom}(\bar{g})\}].$$

3) If $u_1, u_2 \in F_c(\mathcal{T})$, $h : u_1 \rightarrow u_2$ satisfies $[\eta\nu \Leftrightarrow h(\eta)h(\nu)]$ and for $\ell = 1, 2$ we have $\bar{g}^\ell \in F_c(\mathcal{T}, e_2)$, $g_\eta^1 \geq g_{h(\eta)}^2$ (for $\eta \in u_1$), $\bar{\gamma}^\ell = \langle \gamma_\eta^\ell : \eta \in u_\ell \rangle$ is a \triangleleft -decreasing sequence of ordinals, $\gamma_\eta^2 \geq \gamma_{h(\eta)}^2$ and the second player wins in $\mathfrak{D}^{\bar{\gamma}^2}(D, \bar{g}^2, e)$ then the second player wins in $\mathfrak{D}^{\bar{\gamma}^1}(D, \bar{g}^1, e)$.

Proof. 1) The “if part” is trivial, the “only if part” [FILL] is as in [Sh 386].
2), 3) Left to the reader.

The following is a consequence of a theorem of Dodd and Jensen [DoJe81]:

3.13 Theorem. *If λ is a cardinal, $S \subseteq \lambda$ then:*

- (1) $\mathbf{K}[S]$, the core model, is a model of $ZFC + (\forall \mu \geq \lambda) 2^\mu = \mu^+$.
- (2) If in $\mathbf{K}[S]$ there is no Ramsey cardinal $\mu > \lambda$ (or much weaker condition holds) then $(\mathbf{K}[S], \mathbf{V})$ satisfies the μ -covering lemma for $\mu \geq \lambda + \aleph_1$; i.e. if $B \in \mathbf{V}$ is a set of ordinals of cardinality $\leq \mu$ then there is $B' \in \mathbf{K}[S]$ satisfying $B \subseteq B'$ and $\mathbf{V} \models |B'| \leq \mu$.
- (3) If $\mathbf{V} \models (\exists \mu \geq \lambda)(\exists \kappa)[\mu^\kappa > \mu^+ > 2^\kappa]$ then in $\mathbf{K}[S]$ there is a Ramsey cardinal $\mu > \lambda$.

3.14 Lemma. *Suppose*

- (a) \mathbf{n} is a semi-full niceness content thin or medium $\kappa = \aleph_1$
- (b) $f^* \in {}^\kappa \text{Ord}$, $\lambda > \lambda_0 =: \sup\{(2^{| \mathcal{U} / e |}^{\aleph_0}) : e \in \mathbf{e}_\mathbf{n}\}$
- (c) for every $A \subseteq \lambda_0$, in K there is a Ramsey cardindal $> \lambda_0$, then for every filter $D \in \text{FIL}_\mathbf{n}(e)$ is nice to f^* .

Remark. 1) The point in the proof is that via forcing we translate the filters from $\text{FIL}(e, \mathcal{U})$ to normal filters on κ [for higher κ 's cardinal restrictions are better].

2) At present we do not care too much what is the value of λ_0 , i.e., equivalently, how much we like the set S to code.

Saharon: compare with [Sh:g, V], i.e., improve as there! But if we use $\mathbf{e} = \{e\}$, the proofs are more similar to [Sh:g, V] we can consider just $\text{Levy}(\aleph_1, |D|)$, now in some proofs we may consider filters generated by $|\text{pcf}(\mathbf{a})|$ set $|\mathbf{a}| < \aleph_\omega$.

First Proof. Without loss of generality $(\forall i) f(i) \geq 2$. Let $S \subseteq \lambda_0$ be such that $[\alpha < \mu \ \& \ A \subseteq 2^{|\alpha|^{\aleph_0}} \Rightarrow A \in \mathbf{L}[S]], \mathbf{e} \in \mathbf{L}[S]$ (see 3.11(6)) and: if $g \in {}^\kappa \text{Ord}$, $(\forall i <$

$\kappa_1)g(i) \leq f(i)$ then $g \in \mathbf{L}[S]$ (possible as $\prod_{i < \omega_1} |f(i) + 1| \leq \lambda_0$. We work for awhile in $\mathbf{K}[S]$. In $\mathbf{K}[S]$ there is a Ramsey cardinal $\mu > \lambda_0$ (see 3.13(3)). Let in $\mathbf{K}[S]$.
Let

$$Y_0 = \{X : X \subseteq \mu, X \cap \kappa \text{ a countable ordinal } > 0, \{\kappa, \lambda_0\} \subseteq X, \\ \text{moreover } X \cap \lambda_0 \text{ is countable}\}.$$

Let

$$Y_* = Y_1 = \{X \in Y_0 : X \text{ has order type } \geq f(X \cap \kappa)\}.$$

Now for $g \in {}^\kappa\text{Ord}$ such that $\bigwedge_{i < \omega_1} g(i) < f(i)$ let \hat{g} be the function with domain Y_1 , $\hat{g}(X)$ = the $g(X \cap \kappa)$ -th member of X .

Let $D_* = \{A_i : \kappa \leq i \leq 2^{|\mathcal{Y}/e|}\}$ and we arrange $\langle A_i^D : \kappa \leq i < 2^{|\mathcal{Y}/e|} \rangle \in \mathbf{L}[S]$, (as \mathcal{Y}/e has cardinality $< \mu^*$, so $2^{|\mathcal{Y}/e|} \leq \lambda_0$).

Let J be the minimal fine normal ideal on Y (in $\mathbf{K}[S]$) to which $Y \setminus Y_D$ belongs where

$$Y_D = \{X : X \in Y_* \text{ and } i \in (\kappa, 2^{|\mathcal{Y}/e|}) \cap X \Rightarrow X \cap \omega_1 \in A_i\}.$$

Clearly it is a proper filter as $\mathbf{K}[S] \models \text{"}\mu \text{ is a Ramsey cardinal"}$.

3.15 Observation. Assume

- (a) \mathbb{P} is a proper forcing notion of cardinality $\leq |\alpha|^{\aleph_0}$ for some $\alpha < \mu^*$ (or just $\mathbb{P}, \text{MAC}(\mathbb{P}) \in \mathbf{K}[S]$ and $\{X \in Y_1 : X \cap (\text{MAC}(\mathbb{P})) \text{ is countable}\} \in Y_* \text{ mod } J$ where $\text{MAC}(\mathbb{P})$ is the set of maximal antichains of \mathbb{P}) and let $J^\mathbb{P}$ be the normal fine ideal which J generates in $\mathbf{V}^\mathbb{P}$.

- (1) F -positiveness is preserved; i.e. if $X \in \mathbf{K}[S], X \subseteq Y_1, F \in \text{FIL}$ and $\mathbf{V} \models \text{"}X \neq \emptyset \text{ mod } F\text{"}$ then $\Vdash_\mathbb{P} \text{"}X \neq \emptyset \text{ mod } F^\mathbb{P}\text{"}$.
- (2) Moreover, if $\mathbb{Q} \triangleleft \mathbb{P}$, (\mathbb{Q} proper and) \mathbb{P}/\mathbb{Q} is proper then forcing with \mathbb{P}/\mathbb{Q} preserve $F^\mathbb{Q}$ -positiveness.

Continuation of the proof of 3.14.

Case 1: $e = \{e\}$. Here only 3.16(1) is needed and then it is as in the old case.

Case 2: General.

Let $\mathcal{P}(\mathcal{Y}/e) = \{A_\zeta^e : \zeta < 2^{|\mathcal{Y}/e|}\}$.

Now we describe a winning strategy for the second player. In the side we choose also $(p_n, \Gamma_n, f_n), \bar{\gamma}^n, \bar{W}_n$ such that³ (where e_n, A_n are chosen by the second player):

- (A)(i) $\mathbb{P}_n = \prod_{\ell \leq n} \mathbb{Q}_\ell$ where \mathbb{Q}_ℓ is $\text{Levy}(\aleph_1, \mathcal{Y}/e_n)$
 (we could use iterations, too, here it does not matter).
- (ii) $p_n \in \mathbb{P}_n$
- (iii) p_n increasing in n
- (iv) f_n is a \mathbb{P}_n -name of a function from ω_1 to \mathcal{Y}/e_n
- (v) $p_n \Vdash_{\mathbb{P}_n} "f_n(i) \in \mathcal{Y}_i/e_n"$
- (vi) $p_{n+1} \Vdash "f_{n+1}(i) \leq f_n(i) \text{ for every } i < \omega_1"$,
- (vii) f_n is given naturally — it can be interpreted as the generic object of \mathbb{Q}_n except trivialities.
- (B)(i) $\bar{\gamma}^n, \bar{g}^n$ have the same domain, $\gamma_\eta^n < \mu$
- (ii) $p_n \Vdash_{\mathbb{P}_n} "W_n \subseteq Y_D, W_{n+1} \subseteq W_n"$
- (iii) $\bar{\gamma}^n = \bar{\gamma}^{n+1} \upharpoonright \text{Dom}(\bar{\gamma}^n)$, $\text{Dom}(\bar{\gamma}^n) = \text{Dom}(\bar{g}^n)$ and $\bar{\gamma}^n$ is \triangleleft -decreasing
- (iv) $p_n \Vdash_{\mathbb{P}_n} "\{X \in Y_D : \text{for } \ell \in \{0, \dots, n\}, f_\ell(X \cap \omega_1) \in A_\ell \text{ and } \bigwedge_{\eta \in \text{Dom}(\bar{g}^n)} \hat{g}_\eta(X) = \gamma_\eta \text{ and for } \ell \in \{-1, 0, \dots, n-1\}, \zeta \in X \cap 2^{|\mathcal{Y}/e_\ell|} \text{ we have: } A_\zeta^{e_\ell} \in D_\ell \Rightarrow f_\ell(X \cap \omega_1) \in A_\zeta^{e_\ell}\} \supseteq W_n \neq \emptyset \text{ mod } F^{\mathbb{P}_n}"$
- (v) $\bar{g}^n = \bar{g}^{n+1} \upharpoonright \text{Dom}(\bar{g}^n)$ [difference]
- (C)(i) $D_n = \{Z \subseteq \mathcal{Y}/e_n : p_n \Vdash_{\mathbb{P}_n} "\{X \in J_D : f_n(X \cap \omega_1) \notin Z\} = \emptyset \text{ mod } (D_n^{\mathbb{P}_n} + W_n)"\}$
- (ii) \bar{g}^n is D_n -decreasing. [Saharon: diff]

Note that $D_n \in \mathbf{K}[S]$, so every initial segment of the play (in which the second player uses this strategy) belongs to $\mathbf{K}[S]$.

By (B)(iii) this is a winning strategy. □_{3.14}

³For the forcing notions actually used below by the homogeneity of the forcing notion the value of p_n is immaterial

Recall all normal filters on \mathscr{U}/e belong to $\text{FIL}(e)$.

Alternate: We split the proof to a series of claims and definitions.

3.16 Definition. 1) $W_* = \{u \subseteq \mu : \text{otp}(u) \geq f^*(u \cap w_1) \text{ and } u \cap \lambda \text{ is countable}\}$.
 2) Let J be the following ideal on Y_0 :
 $W \in J$ iff for some model M on μ with countable vocabulary (with Skolem function) we have

$$W_* \supseteq W \subseteq \{w \in W_* : w = c\ell_M(w)\}.$$

3) For $g \in \prod_{i < \kappa} (f(i) + 1)$ let \hat{g} be the function with domain Y_* and $\hat{g}(A)$ is the $g(i)$ -th member of A .
 4) For $W \in J^+$ let $\text{proj}(W) = \{A \subseteq w_1 : \{w \in W : w \cap w_1 \notin A\} \in J\}$.

3.17 Fact. 1) $Y_* \notin J$.
 2) J is a fine normal filter on W_* (and $W_* \notin J$) in fact the ideal of non-stationary subsets of W_* .
 3) $Y_{\bar{A}} \in J^+$ if $\bar{A} = \langle A_i : i < 0 \rangle$, 2^{\aleph_1} list the subset of some normal filter D on ω_1 (see 3.23's proof).
 4) If \bar{A}', \bar{A}'' list the same normal filter on w_1 then $Y_{\bar{A}'} = Y_{\bar{A}''} \bmod J$.
 5) For $g \in \prod_{i < \omega} (f^*(i) + 1)$, \hat{g} is well defined, is a choice function of Y_* .
 6) If $g_1 <_D g_2$ then $\hat{g}_1 \upharpoonright J_D < \hat{g}_2 \upharpoonright J_D \bmod J + Y_*$.

Proof. 1) As μ is a Ramsey cardinal $> \lambda_0$.

2) By the definitions.

3) Easy.

3.18 Claim. Assume \mathbb{Q} is an \aleph_1 -complete forcing notion with $\leq \lambda_0$ maximal antichains.

1) Forcing with \mathbb{Q} preserves all our assumptions:

- (a) μ is a Ramsey cardinal⁺
- (b) W_* is a family of subsets of μ such that $\text{otp}(w) \geq f(w \cap \omega_1)$ and J , defined above, is a fine normal ideal on Y_* satisfying 3.17(3)...then we can forget (a).

2) Forcing with \mathbb{Q} preserves “ $y \in J^+$ ” (i.e. if $W \in J^+$ then $\Vdash_{\mathbb{Q}} “W \in J^+”$).

Proof. Easy, fill.

3.19 Definition. Assume $e \in \mathbf{e}_n$ and $D \in \text{FIL}_n(e)$.

1) $\mathbb{Q} = \mathbb{Q}_e = \{f : f \text{ is a function with domain a countable ordinal such that } i \in \text{Dom}(f) \Rightarrow f(i) \in \mathcal{Y}_i^n\}$.

2) f_e is the \mathbb{Q} -name $\cup\{f : f \in G_{\mathbb{Q}_e}\}$.

3) Let D/f_e be the \mathbb{Q}_e -name of $\{A \subseteq \omega_1 : \text{for every } B \in D \text{ for stationarily many } i < \omega_1, f_e(i) \in B\}$ and $\text{nor}(D, f_e)$ the normal filter which D/f_e generates.

4) For $W \in J^+$ let $\text{lift}(W, D) = \{A \subseteq \mathcal{Y}/e \text{ for some } B \in D : \Vdash_{\mathbb{Q}_e} “\{w \in W : f_e(w \cap \omega_1) \in B \setminus A \in J”$ (note that we have enough homogeneity for \mathbb{Q}_e).

3.20 Claim. Assume $e \in \mathbf{e}_n$ and $D \in \text{FIL}_n(e)$.

1) $\Vdash_{\mathbb{Q}} “D/f_e \text{ is a normal filter on } \omega_1”$, (i.e. $w_1 \notin D$).

2) $|\mathbb{Q}_e| \leq |\mathcal{Y}^n/e|^{\aleph_0}$ so $Z^{|\mathbb{Q}_e|} \leq \lambda_0$ hence \mathbb{Q}_e has $\leq \lambda_0$ maximal antichains; in fact, equality holds as we have demand $|\mathcal{Y}/e| = |\cup\{\mathcal{Y}_i : i \in [i_0, \omega_1)\}|/e|$ for every $e \in \mathbf{e}$.

3) Combine scite 3.2A(4) + 3.19 - FILL.

3.21 Definition. 1) We say that $\mathfrak{x} = (e, D, \bar{g}, \bar{\alpha}, f, W)$ is a good position (in the content of proving 3.14) if

(a) $e \in \mathbf{e}_n$

(b) $D \in \text{FIL}_n(e)$

(c) $\bar{g} = \langle g_\eta : \eta \in u \rangle \in \text{Fc}(\mathcal{T}, e)$, so $u = u^{\mathfrak{x}}$

(d) $\bar{\alpha} = \langle \alpha_\eta : \eta \in u \rangle, \alpha_\eta < \mu$

(e) $p \in \mathbb{Q}_e$

(f) $W = \{w \in W^* : \hat{g}_\eta(w) = \alpha_\eta \text{ for } \eta \in u\} \in J^+$

(g) $p \Vdash_{\mathbb{Q}_e} “W^{\mathfrak{x}} \cap W_{D, f_e} \in J^+”$ and $\text{proj}(W^{\mathfrak{x}} \cap W_{D, f_e}) = D \text{ nor}(D, f_e)$ [FILL].

3.22 Observation. 1) If $\mathfrak{x} = (e, D, \bar{g}, \bar{\alpha}, p, W)$ is a good position then

(a) $\bar{\alpha}$ is decreasing

(b) D_W .

3.23 Claim. *If $e \in \mathbf{e}_n$, $D \in \text{FIL}_n(e)$ and $\bar{g} = \langle g_\eta : \eta \in u \rangle \in \text{Fc}(\mathcal{T}, e)$ and $g_\eta \leq f[e]$ for every $\eta \in \text{Dom}(\bar{g})$ then we can find a good position \mathfrak{x} with $\bar{g}^\mathfrak{x} = e^\mathfrak{x} = e$, $\bar{g}^\mathfrak{x} = g$ and $D \subseteq D^\mathfrak{x}$.*

Proof. Let $\mathbf{G} \in \mathbb{Q}_e$ be generic over \mathbf{V} and $f_e = f_e[G]$. So in $\mathbf{V}[\mathbf{G}]$ the set $W_{D, f_e[\mathbf{G}]}$ belongs to J^+ (by 3.17(3)), i.e., let $\langle A_\zeta^{D_1} : \zeta < \zeta^* \rangle$ list D_1 and $W, D, f_e = \{w \in W : \text{if } \zeta \in w \cap \zeta^* \text{ then } f_e(i) = f_e[\mathbf{G}](i) \in A_\zeta\}$.

Also \hat{g}_η defined in 3.16(3) is a choice function on W_{D, f_e} (see 3.17(4)), so as J is a normal ideal and u finite, we can find $\bar{\alpha} = \langle \alpha_\eta : \eta \in u \rangle$ such that $W = \{w \in W_{D, f_e} : \hat{g}_\eta(w) = \alpha_\eta \text{ for } \eta \in u\}$ belongs to J^+ . As all this holds in $\mathbf{V}[\mathbf{G}]$. So $\bar{\alpha}$ there is a condition $p \in \mathbb{Q}_e$ which forces this, and we are done.

3.24 Claim. *Assume that*

- (a) $\mathfrak{x}_1 = (e_1, D_1, \bar{g}_1, \bar{\alpha}_1, p, W_1)$ is a good position
- (b) $\bar{g}_2 = \langle g_\eta^2 : \eta \in u_2 \rangle \in \text{Fc}(\mathcal{T}, \mathbf{n})$ and $\bar{g}_2 \restriction u_1 = \bar{g}_1$
- (c) $e_1 \leq e_2$ in \mathbf{e}_n and $D_2 \in \text{FIL}_n(e_2)$ or just $\mathcal{A} \subseteq \mathcal{P}(\mathcal{Y}_n/e_2)$, $\mathcal{A} = \{A_\zeta : \zeta < \zeta^*\}$
- (d) $p_1 \Vdash_{\mathbb{Q}_{e_1}} \text{“}\{w \in W_1 : \mathcal{Y}_{w \cap w_1} \not\subseteq \cup\{A_\zeta : \zeta \in \zeta^* \cap w\}\} \text{ does not belong to } J^{\mathbf{V}[\mathbb{Q}_{e_1}]}\text{”}.$

Then we can find a good position \mathfrak{x}_2 such that $e^{\mathfrak{x}_2} = e_2$, $\bar{g}^{\mathfrak{x}_2} = \bar{g}^2$ and $D_2 \subseteq D^{\mathfrak{x}_2}$.

Proof. Let \mathbf{G} be a subset of $\mathbb{Q}_{e_1[\mathfrak{x}_1]}$ generic over \mathbf{V} such that $p^{\mathfrak{x}_1} \in \mathbf{G}_1$. Now \mathbb{Q}_{e_2} is an \aleph_1 -complete forcing of cardinality $\leq |\mathcal{Y}_n/e_2|^{\aleph_0} \leq \lambda_0$ and \mathbb{Q}_{e_1} is \aleph_1 -complete $|\mathbb{Q}_{e_1}| \leq |\mathcal{Y}_n/e_1|^{\aleph_0} \leq |\mathcal{Y}_n/e_2|^{\aleph_0} \leq \lambda_0$, so \mathbb{Q}_{e_2} satisfies the same conditions in $\mathbf{V}[\mathbf{G}_1]$ (if λ_0 is no longer a cardinal it does not matter).

Note that by assumption (c)

- ⊗ in $\mathbf{V}[\mathbf{G}_1], \mathbb{Q}_{e_2} \Vdash \text{“the set } \{W_2^1 =: \{w \in W_1[\mathbf{G}_1] : \text{the set } ((f_{e_1}[\mathbf{G}_1])(w \cap \omega_1))^{[e_2]} \in \mathcal{Y}_{w \cap \omega_1}/e_2 \text{ is not included in } \cup\{A_\zeta : \zeta \in w\}\} \text{ is stationary (i.e. } \notin J)\text{”}.$

We continue as in the previous claim.

3.25 Claim. *If clauses (a) + (b) of 3.23 holds, then a sufficient condition for clause (c) is*

(c)' FILL.

3.26 Proof of 3.14. During the play, the player II chooses also a good position \mathfrak{r}_n and maintains $\bar{g}^{\mathfrak{r}_n} = \bar{g}_n, \bar{\alpha}^{\mathfrak{r}_n} = \bar{\alpha}$.

3.27 Remark. 1) From the proof, instead $\mathbf{K}[S] \models \text{“}\lambda \text{ is Ramsey”}$, $\mathbf{K}[S] \models \text{“}\mu \rightarrow (\alpha)_{\lambda_0}^{<\omega} \text{ for } \alpha < \lambda_0\text{”}$ is enough for showing for 3.14.

2) Also if $\prod_{i < \omega_1} (|f(i)| + 1) < \mu_0, [\alpha < \mu_0 \Rightarrow |\alpha|^{\aleph_0} < \mu_0]$, it is enough: $S \subseteq \alpha < \mu_0 \Rightarrow$ in $\mathbf{K}[S]$ there is $\mu \rightarrow (\alpha)_2^{<\omega}$.

3.28 Theorem. *Assume \mathbf{n} is a κ -niceness context. Let $D^* \in \text{FIL}(e, \mathcal{Y})$ be a normal ideal on $\mathcal{Y}_{\mathbf{n}}/e$. If for every $f : \mathcal{Y} \rightarrow (\sup\{\text{Suc}(D') : D' \in \text{FIL}_{\mathbf{n}}\})^+$ supported by some $e \in \mathbf{e}_{\mathbf{n}}$. $D_{\mathbf{n}}^*$ is nice to f , then for every $f \in {}^\kappa\text{Ord}$, \mathbf{n} is nice to f .*

Proof. By determinacy of the games (and the LS argument).

3.29 Remark. 0) The value $|\text{FIL}_{\mathbf{e}}|$ really should be an upper bound.

1) So, the existence of $\mu, \mu \rightarrow (\alpha)_{\aleph_0}^{<\omega}$ for every $\alpha < (\sum_{\chi < \mu} \chi^\kappa)^+$, is enough for “ D^* is nice”.

2) If there is a nice D 's in the plays from 3.7, the second player winning strategy can be chosen such that all subsequent filters are nice: just by renaming have $g_{<}$ constant large enough. [Saharon: diff]

3.30 Claim. *In claim 3.14 we can omit “ $\kappa_{\mathbf{n}} = \aleph_1$ ”.*

Proof. Let $\mathbb{P} = \text{Levy}(\aleph_0, \kappa_{\mathbf{n}})$. Now

(*) also in $\mathbf{V}^{\mathbb{P}}$ the object \mathbf{n} is a successor content, if we do not distinguish between $D \in \text{FIL}_{\mathbf{n}}$ and $\{A \in \mathbf{V}^{\mathbb{P}} : A \subseteq \mathcal{Y}/e(D) \text{ and } (\exists B \in D)(B \subseteq A)\}$.

3.31 Conclusion.: Let $\lambda_0 = (\sup\{|\text{Suc}_{\mathbf{n}}(D')| : D' \in \text{FIL}_{\mathbf{n}}\})^+ \cup \{2^{|\mathcal{Y}/e|^{<\kappa}} : e \in \mathbf{e}_{\mathbf{n}}\}^+, \mu^* \geq \aleph_2$; if for every $S \subseteq \lambda_0$ there is a Ramsey cardinal in $\mathbf{K}[S]$ above λ_0 then \mathbf{n} is nice.

Proof. By 3.14, 3.28.

3.32 Concluding Remark. 1) We could have used other forcing notions, not $\text{Levy}(\kappa, |\mathcal{Y}/e_n|)$. E.g., if $\kappa = \aleph_1, \mu = \kappa^+$ we could use finite iterations of the forcing of Baumgartner to add a club of ω_1 , by finite conditions. (So this forcing notion has cardinality \aleph_1).

Then in 3.14 we can weaken the demands on λ_0 : $\lambda_0 = \sum_{\chi < \mu_0} 2^\chi + \prod_{i < \omega_1} |1 + f(i)| + |\mathbf{e}|$,

hence also in 3.31, $\lambda_0 = \sum_{\chi < \mu_*} 2^\chi$ is O.K.

2) Concerning $|\mathbf{e}|$ remember 3.11(5),(6).

3) Similarly to (1). If $\theta < \mu \Rightarrow \text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$ then by 2.6 we can use forcing notions of Todorćević for collapsing $\theta < \mu$ which has cardinality $< \mu$.

4) If we want to have $\lambda_0 =: \prod_{i < \omega_1} |f(i) + 2|$ (or even $T_D(f + 2)$), we can get this

by weakening further the first player letting him choose only A_n which are easily definable from the \bar{g}^{n-1} , we shall return to it in a subsequent paper.

§4 RANKS

4.1 Convention. 1) Like 3.2 and:

2) $\bar{g}^* \in F_c(\mathcal{T}, e^*, \mathcal{Y}), \eta^* \in \text{Dom}(\bar{g}^*), \nu^*$ an immediate successor of η^* not in $\text{Dom } g^*, D^* \in \text{FIL}(e^*, \mathcal{Y})$ is such that in $\mathfrak{D}^{\bar{\gamma}^*}(D^*, \bar{g}^*, e^*)$ second player wins (all constant for this section). $\text{FIL}^*(e)$ will be the set of $D \in \text{FIL}(e, \mathcal{Y})$ such that $e \geq e^*, (D^*)^{[e]} \subseteq D$ and in $\mathfrak{D}^{\bar{\gamma}^*}(D^*, \bar{g}^*, e^*)$ second player wins. (So actually $\text{FIL}(e^*, \mathcal{Y})$ depends on D^*, \bar{g}^*, e^* , too).

4.2 Definition. 1) $\text{rk}_D^5(f)$ for $D \in \text{FIL}^*(e, \mathcal{Y}), f \in \mathcal{Y}/e\text{Ord}, f <_D \bar{g}_{\eta^*}^*$ will be: the minimal ordinal α such that for some $D_1, e_1, \bar{\gamma}^1$ we have $D^{[e_1]} \subseteq D_1 \in \text{FIL}(e_1, \mathcal{Y}), \bar{\gamma}^1 = \bar{\gamma}^* \hat{\wedge} \langle \nu^*, \alpha \rangle$ (i.e. $\text{dom}(\bar{\gamma}^1) = (\text{dom}(\bar{\gamma}^*)) \cup \{\nu^*\}, \bar{\gamma}^1 \upharpoonright \text{dom}(\bar{\gamma}^*) = \bar{\gamma}^*, \gamma_{\nu^*}^1 = \alpha$) and in $\mathfrak{D}^{\bar{\gamma}^1}(D, \bar{g}^* \hat{\wedge} \langle \nu^*, f \rangle)$ second player wins and ∞ if there is no such α .
2) $\text{rk}_D^4(f)$ is $\sup\{\text{rk}_{D+A}^5(f) : A \in D^+\}$.

4.3 Claim. 1) $\text{rk}_D^5(f)$ is (under the circumstances of 4.1, 4.2) an ordinal $< \gamma_{\eta^*}^*$.
2) $\text{rk}_D^4(f)$ is an ordinal $\leq \gamma_{\eta^*}^*$.

4.4 Claim. If $D \in \text{FIL}^*(e, \mathcal{Y}), h <_D f <_D g_{\eta^*}^*$ then $\text{rk}_D^5(h) < \text{rk}_D^5(f)$.

Proof. Let e_1, D_1 witness $\text{rk}_D^5(f) = \alpha$ so $e(D) \leq e_1, D \subseteq D_1 \in \text{FIL}^*(e_1)$ and in $G^{\bar{\gamma}^* \hat{\wedge} \langle \nu^*, \alpha \rangle}(D_1, \bar{g}^* \hat{\wedge} \langle \nu^*, f \rangle, e)$ second player wins. We play for the first player: $e = e_1, A_0 = \mathcal{Y}/e_1, \bar{g}^0 = \bar{g}^* \hat{\wedge} \langle \nu^*, f \rangle \hat{\wedge} \langle \nu^* \hat{\wedge} \langle 0 \rangle, g \rangle$, now the first player should be able to answer say $e_2, D_2, \bar{\gamma}^2$. So $\gamma_{\nu^* \hat{\wedge} \langle 0 \rangle}^2 < \gamma_{\nu^*}^2 = \alpha$, and by 3.12(3), we know that in $G^{\bar{\gamma}'}(D_2, \bar{g}^* \hat{\wedge} \langle \nu^*, g \rangle, e_2)$ where $\bar{\gamma}' = \bar{\gamma}^* \hat{\wedge} \langle \nu^*, \gamma_{\nu^* \hat{\wedge} \langle 0 \rangle}^2 \rangle$, second player wins. $\square_{4.4}$

4.5 Claim. Let $e \geq e^*, D \in \text{FIL}^*(e, \mathcal{Y})$.

1) For $e \geq e(D), A \in (D^{[e]})^+, f \in \mathcal{Y}/e\text{Ord}, f <_D g_{\eta^*}^*$ we have:

$$\text{rk}_D^5(f) \leq \text{rk}_{D^{[e]}+A}^5(f) \leq \text{rk}_{D^{[e]}+A}^4(f) \leq \text{rk}_D^4(f).$$

2) If $e_2 \geq e_1 \geq e(D), f_\ell \in \mathcal{Y}\text{Ord}$ is supported by $e_\ell, f_1 \leq_D f_2 <_D g_{\eta^*}^*$ then $\text{rk}_D^\ell(f_1) \leq \text{rk}_D^\ell(f_2)$ for $\ell = 4, 5$.

Proof. Left to the reader.

§5 MORE ON RANKS AND HIGHER OBJECTS

5.1 Convention.

- (a) μ^* is a cardinal $> \aleph_1$ (using \aleph_1 rather than an uncountable regular κ is to save parameters)
- (b) \mathcal{Y} a set of cardinality $\sum_{\kappa < \mu^*} \kappa$
- (c) ι a function from \mathcal{Y} onto ω_1 , $|\iota^{-1}(\{\alpha\})| = |\mathcal{Y}|$ for $\alpha < \omega$,
- (d) Eq the set of equivalence relation e on \mathcal{Y} such that:
 - (α) $yez \Rightarrow \iota(y) = \iota(z)$
 - (β) each equivalence class has cardinality $|\mathcal{Y}|$
 - (γ) e has $< \mu^*$ equivalence classes
- (e) D denotes a normal filter on some \mathcal{Y}/e ($e \in \text{Eq}$), we write $e = e(D)$. The set of such D 's is $\text{FIL}(\mathcal{Y})$.
- (f) E denotes a set of D 's as above, such that:
 - (α) for some $D = \text{Min } E \in E$ ($\forall D'$) [$D' \in E \Rightarrow (e, D) \leq (e(D'), D')$]
 - (β) if $D \in E$, $A \subseteq \mathcal{Y}/e_1$, $e_1 \geq e(D)$, $A \neq \emptyset \text{ mod } D$ then $D^{[e_1]} + A \in E$
- (g) $E^{[e]} =: \{D \in E : e(D) = e\}$
- (h) \mathcal{E} denotes a set of E 's as above, such that:
 - (α) there is $E = \text{Min } \mathcal{E} \in \mathcal{E}$ satisfying $(\forall E') (E' \in \mathcal{E} \Rightarrow E' \subseteq E)$
 - (β) if $D \in E \in \mathcal{E}$ then $E_{[D]} = \{D' : D' \in E \text{ and } (e(D), D) \leq (e(D'), D')\} \in \mathcal{E}$.

5.2 Definition. 1) We say E is λ -divisible when: for every $D \in E$, and Z , a set of cardinality $< \lambda$ there is D 's such that:

- (α) $D' \in E$
- (β) $(e(D), D) \leq (e(D'), D')$
- (γ) $\mathbf{j} : \mathcal{Y}/e(D') \rightarrow Z$
- (δ) for every function $h : \mathcal{Y}/e(D) \rightarrow Z$ we have $\{y/e(D') : h(y/e(D)) = (y/e(D'))\} \neq \emptyset \text{ mod } D'$.

2) We say E has λ -sums when: for every $D \in E \in \mathcal{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda \rangle$ of subsets of $\mathcal{Y}/e(D)$ there is $Z^* \subseteq \mathcal{Y}/(e/(D))$, such that: $Z^* \cap Z_\zeta = \emptyset \bmod D$ and: [if $(e(D), D) \leq (e', D')$, $e' = e(D')$, $D' \in E_{[D]}$ and $\bigwedge_{\zeta} Z_\zeta^{[e']} = \emptyset \bmod D'$ then

$Z^* \in D'$].

3) We say E has weak λ -sum if for every $D \in E \in \mathcal{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda \rangle$ of subsets of $\mathcal{Y}/e(D)$ there is $D^*, D^* \in E_{[D]}$ such that:

- (α) if $(e(D), D) \leq (e', D')$, $D' \in E_{[D]}$ and $Z_\zeta = \emptyset \bmod D'$ for $\zeta < \zeta^*$ and $e(D^*) \leq e(D')$ then $D^* \subseteq D'$ (more exactly $D^{[*]} \subseteq D^{[*]}$ and)
- (β) $Z_\zeta = \emptyset \bmod D^*$ for $\zeta < \zeta^*$.

4) If $\lambda = \mu^*$ we omit it. We say \mathcal{E} is λ -divisible if every $E \in \mathcal{E}$ has. We say \mathcal{E} has weak λ -sums if: [rest diff] for every $E \in \mathcal{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda \rangle$ of subsets of $\mathcal{Y}/e(E)$ there is $E^*, E^* \in \mathcal{E}_{[E]}$ such that:

- (α) if $(e(E), E) \leq (e', E')$, $E' \in \mathcal{E}$ and $Z_\zeta = \emptyset \bmod \text{Min}(E')$ for $\zeta < \zeta^*$ and $e(E^*) \leq e(E')$ then $E^* \subseteq E'$
- (β) $Z_\zeta = \emptyset \bmod \text{Min}(E^*)$ for $\zeta < \zeta^*$.

We now define variants of the games from §3.

5.3 Definition. For a given \mathcal{E} , for every $E \in \mathcal{E}$:

1) We define a game $G_2^*(E, \bar{g})$. In the n -th move first player chooses $D_n \in E_{n-1}$ (stipulating $E_{-1} = E$) and choose $\bar{g}_n \in F_c(\omega, e(D_n), \mathcal{Y})$ extending \bar{g}_{n-1} (stipulating $\bar{g}_{-1} = \bar{g}$) such that \bar{g}_n is D_n -decreasing. Then the second player chooses E_n , $(E_{n-1})_{[D_n]} \subseteq E_n \in \mathcal{E}$.

In the end the second player wins if $\bigcup_{n < \omega} \text{Dom } \bar{g}_n$ has no infinite branch.

2) We define a game $G_2^{\bar{\gamma}}(E, \bar{g})$ where $\text{Dom}(\bar{\gamma}) = \text{Dom}(\bar{g})$, each γ_η an ordinal, $[\eta < \nu \Rightarrow \gamma_\eta > \gamma_\nu]$ similarly to $G_2^*(E, \bar{g})$ but the second player in addition chooses an indexed set $\bar{\gamma}_n$ of ordinals, $\text{Dom}(\bar{\gamma}_n) = \text{Dom}(\bar{g}_n)$, $\bar{\gamma}_n \upharpoonright \text{Dom}(\bar{\gamma}_{n-1}) = \bar{\gamma}_{n-1}$ and $[\eta < \nu \Rightarrow \gamma_{n,\eta} > \gamma_{n,\nu}]$.

5.4 Definition. 1) We say \mathcal{E} is nice to $\bar{g} \in F_c(\mathcal{T}, e, \mathcal{Y})$ if for every $E \in \mathcal{E}$ with $e \leq e(E)$ the second player wins the game $\mathcal{D}_2^*(E, \bar{g})$.

2) We say \mathcal{E} is nice if it is nice to \bar{g} whenever $E \in \mathcal{E}$, $e \leq e(E)$, $\bar{g} \in F_c(\mathcal{T}, e)$, \bar{g} is $(\text{Min } E)$ -decreasing, we have: $\mathcal{E}_{[E]}$ is nice to \bar{g} .

3) If $\text{Dom}(\bar{g}) = \{<\>\}$ we write $g_{<\>}$ instead \bar{g} .

4) We say \mathcal{E} is nice to α if it is nice to the constant function α .

- 5.5 Claim.** 1) If \mathcal{E} is nice to f , $f \in F_c(\mathcal{T}, e, \mathcal{Y})$, $g \in F_c(\mathcal{T}, e, \mathcal{Y})$, $g \leq f$ then \mathcal{E} is nice to f .
 2) The games from 5.4 are determined, and the winning side has winning strategy which does not need memory.
 3) The second player wins $G_2^*(E, \bar{g})$ iff for some $\bar{\gamma}$ second player wins $G_2^{\bar{\gamma}}(E, g)$.
 4) If the second player wins $G_2^{\bar{\gamma}}(E, f)$, $\bar{g} \in F_c(\mathcal{T}, e(E))g_\eta \leq f$ for $\eta \in \text{Dom}(\bar{g})$ then the second player wins in $G_2^{\bar{\gamma}}(E, \bar{g})$ when we let

$$\gamma_\eta = \gamma + \max\{(\ell g(\nu) - \ell g(\eta) + 1) : \nu \text{ satisfies } \eta \trianglelefteq \nu \in \text{Dom}(\bar{g})\}.$$

5.6 Lemma. Suppose $f_0 \in {}^{(\mathcal{Y}/e)}\text{Ord}$, $e \in \text{Eq}$ and $\lambda_0 =: \sup\{\prod_{x \in Y} \mathcal{Y}_e(f_0^{[e]}(x) + 1) : e \text{ satisfies } e_0 \leq e \in \mathbf{e}\}$.

- 1) If there is a Ramsey cardinal $\geq \cup\{f(x) + 1 : x \in \text{Dom}(f_0)\}$ then there is a μ^* -divisible \mathcal{E} nice to f_0 having weak μ^* -sums.
 2) If for every $A \subseteq \lambda_0$ there is in $\mathbf{K}[A_0]$ a Ramsey cardinal $> \lambda_0$, then there is a μ^* -divisible \mathcal{E} which has weak μ^* -sums and is nice to f .
 3) In part 2 if $\lambda_0 = 2^{<\mu_0}$ then there is a μ^* -divisible nice \mathcal{E} which has weak μ^* -sums.

5.7 Remark. This enables us to pass from “pp $_{\Gamma(\theta, \aleph_1)}$ large” to “pp $_{\text{normal}}$ is large”.

Proof. 1) Define $f_1 \in {}^{(\aleph_1)}\text{Ord}$, $f_1(i) = \sup\{f_0(y/e) : \iota(y) = i\}$, let λ be such that: $\lambda \rightarrow (\sup\{f_1(i)\}_2^{<\omega} : i < \aleph_1)$ (or just $\emptyset \notin D_n^*$ - see below) let $\lambda_n = (\lambda^{\mu^*})^{+n}$,

$$I_n = \{s : s \subseteq \lambda_n, s \cap \omega_1 \text{ a countable ordinal}\}$$

$$J_n = \{s \in I_n : s \cap \lambda \text{ has order type } \geq f_0(s \cap \omega_1)\}.$$

Let D_n^* be the minimal fine normal filter on J_n .

Let for $n < \omega$ and $e \in \text{Eq}$, $H_{n,e} = \{h : h \text{ a function from } J_n \text{ into } \mathcal{Y}/e \text{ such that } \iota(h(s)) = s \cap \omega_1\}$.

Let $\mathbb{P}_n = \{p : p \subseteq J_n, p \neq \emptyset \text{ mod } D_n^*\}$, $\mathbb{P} = \bigcup_{n < \omega} \mathbb{P}_n$ and for $p \in \mathbb{P}$ let $n(p)$ be the

unique n such that $p \in \mathbb{P}_n$.

Let $p \leq q$ (in \mathbb{P}) if $n(p) \leq n(q)$ and $\{s \cap \lambda_{n(p)} : s \in q\} \subseteq p$.

Now for every $e \in \text{Eq}$, $n < \omega$, $p \in \mathbb{P}_n$, $h \in H_{n,e}$ we let:

$$D_p^{n,e,h} = \{A \subseteq \mathcal{Y}/e : h^{-1}(A) \supseteq p \text{ mod } D_{n(p)}^*\}$$

$$E_p^{n,e,h} = \{D_q^{n^1,e^1,h^1} : p \leq q \in P, n^1 = n(q) \text{ and } (n^1, e^1, h^1) \geq (n, e, h)\}$$

where $(n^1, e^1, h^1) \geq (n, e, h)$ means: $n \leq n^1 < \omega$, $e \leq e^1 \in \text{Eq}$, $h^1 \in H_{n^1, e^1}$ and for $s \in J_{(n^1)}$, $h^1(s)^{[e]} = h(s \cap \lambda_n)$ and we define $(p^1, n^1, e^1, h^1) \geq (p, n, e, h)$ similarly. Let

$$\mathcal{E}_p^{n,e,h} = \{E_q^{n^1,e^1,h^1} : p \leq q \in P, n^1 = n(q), (n^1, e^1, h^1) \geq (n, e, h)\}.$$

Note: $(p^1, n^1, e^1, h^1) \geq (p, e, n, h)$ implies $D_{p^1}^{n^1,e^1,h^1} \supseteq D_p^{n,e,h}$, $E_{p^1}^{n^1,e^1,h^1} \subseteq E_p^{n,e,h}$ and $\mathcal{E}_{p^1}^{n^1,e^1,h^1} \subseteq \mathcal{E}_p^{n,e,h}$. Now any $\mathcal{E} = \mathcal{E}_p^{n,e,h}$ ($p \in P$) is as required.

A new point is “ \mathcal{E} is μ^* -divisible”. So suppose $E \in \mathcal{E} = \mathcal{E}_p^{n,e,h}$ so $E = E_q^{n^1,e^1,h^1}$ for some $(q, n^1, e^1, h^1) \geq (p, n, e, h)$. Let Z be a set of cardinality $< \mu^*$, so $(\lambda_{n^1})^{|Z|} = \lambda_{n^1}$; let $\{h_\zeta : \zeta < \zeta^* = |\mathcal{Y}/e_1|^{|Z|} \leq 2^\mu \leq \lambda_{n^1}\}$ list all function h from \mathcal{Y}/e_1 to Z . Let $\langle S_\zeta : \zeta < |\mathcal{Y}/e_1|^{|Z|} \rangle$ list a sequence of pairwise disjoint stationary subsets of $\{\delta < \lambda_{n^1+1} : \text{cf}(\delta) = \aleph_0\}$. Let $e_2 \in \text{Eq}$ be such that $e_1 \leq e_2$ and for every $y \in \mathcal{Y}$, $\{z/e_2 : ze_1y\} = \{x(y/e, t) : t \in Z\}$, we let $q_2, q \leq q_2 \in P$ be: $q_2 = \{s \in J_{n^1+1} : s \cap \lambda_{n^1} \in q \text{ and } \sup s \in \bigcup_\zeta S_\zeta\}$, lastly we define $h^2 : J_{n^1+1} \rightarrow \mathcal{Y}/e_1$ by:

$h^2(s) = x(h^1(s \cap \lambda_{n^1}), h_\zeta(s \cap \lambda_{n^1}))$ if $s \in q_2, \sup s \in S_\zeta$ (for $s \in J_{n^1+1} \setminus q_2$ it does not matter). The proof that q_2, e_2, h^2 are as required is as in [RuSh 117] and more specifically [Sh 212]. As for proving “ $\mathcal{E}_p^{n,e,h}$ has weak μ^* -sums” the point is that the family of fine normal filters on μ has μ^* -sum.

2) Similar to 3.14 (and 3.11(5),(6)).

3) Similar to [Sh 386, 1.7].

□_{5.6}

§6 HYPOTHESES: WEAKENING OF GCH

We define some hypotheses; except the first we do not know now whether their negations are consistent with ZFC.

6.1 Definition. We define a series of hypothesis:

- (A) $\text{pp}(\lambda) = \lambda^+$ for every singular λ .
- (B) If \mathfrak{a} is a set of regular cardinals, $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$ then $|\text{pcf}(\mathfrak{a})| \leq |\mathfrak{a}|$.
- (C) If \mathfrak{a} is a set of regular cardinals, $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$ then $\text{pcf}(\mathfrak{a})$ has no accumulation point which is inaccessible (i.e. λ inaccessible $\Rightarrow \sup(\lambda \cap \text{pcf}(\mathfrak{a})) < \lambda$).
- (D) For every λ , $\{\mu < \lambda : \mu \text{ singular and } \text{pp}(\mu) \geq \lambda\}$ is countable.
- (E) For every λ , $\{\mu < \lambda : \mu \text{ singular and } \text{cf}(\mu) = \aleph_0 \text{ and } \text{pp}(\mu) \geq \lambda\}$ is countable.
- (F) For every λ , $\{\mu < \lambda : \mu \text{ singular of uncountable cofinality, } \text{pp}_{\Gamma(\text{cf}(\mu))}(\mu) \geq \lambda\}$ is finite.
- (D) $_{\theta, \sigma, \kappa}$ For every λ , $\{\mu < \lambda : \mu > \text{cf}(\mu) \in [\sigma, \theta) \text{ and } \text{pp}_{\Gamma(\theta, \sigma)}(\mu) \geq \lambda\}$ has cardinality $< \kappa$.
- (A) $_{\Gamma}$ If $\mu > \text{cf}(\mu)$ then $\text{pp}_{\Gamma}(\mu) = \mu^+$ (or in the definition of $\text{pp}_{\Gamma}(\mu)$ the supremum is on the empty set).
- (B) $_{\Gamma}$, (C) $_{\Gamma}$ Similar versions (i.e. use pcf_{Γ}).

We concentrate on the parameter free case.

6.2 Claim. : In 6.1, we have:

- (1) $(A) \Rightarrow (B) \Rightarrow (C)$
- (2) $(A) \Rightarrow (D) \Rightarrow (E), (A) \Rightarrow (F)$
- (3) $(E) + (F) \Rightarrow (D) \Rightarrow (B)$. [Last implication — by the localization theorem [Sh 371, §2]]
- (4) if $(\forall \mu)(\mu > \text{cf}(\mu) = \aleph_0 \text{ the hypothesis (A) of 6.1 holds.}$
[Why? By [Sh:g, xx].]

6.3 Theorem. Assume Hypothesis 6.1(A).

1) For every $\lambda > \kappa$,

$$\text{cov}(\lambda, \kappa^+, \kappa^+, 2) = \begin{cases} \lambda^+ & \text{if } \text{cf}(\lambda) \leq \kappa \\ \lambda & \text{if } \text{cf}(\lambda) > \kappa. \end{cases}$$

2) For every $\lambda > \kappa = \text{cf}(\kappa) > \aleph_0$, there is a stationary $S \subseteq [\lambda]^{\leq \kappa}$, $|S| = \lambda^+$ if $\text{cf}(\lambda) \leq \kappa$ and $|S| = \lambda$ if $\text{cf}(\lambda) > \kappa$.

- 3) For μ singular, there is a tree with $\text{cf}(\mu)$ levels each level of cardinality $< \mu$, and with $\geq \mu^+(\text{cf}(\mu))$ -branches.
- 4) If $\kappa \leq \text{cf}(\mu) < \mu \leq 2^\kappa$ then there is an entangled linear order \mathcal{T} of cardinality μ^+ .

Proof. 1) By [Sh 400, §1].

2) By part (1) and 2.6.

3, 4) By [Sh 355, §4].

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